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Duality and braiding in twisted quantum field theory

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Abstract: We re-examine various issues surrounding the definition of twisted quantum field theories on flat noncommutative spaces. We propose an interpretation based on nonlocal commutative field redefinitions which clarifies previously observed properties such as the formal equivalence of Green's functions in the noncommutative and commutative theories, causality, and the absence of UV/IR mixing. We use these fields to define the functional integral formulation of twisted quantum field theory. We exploit techniques from braided tensor algebra to argue that the twisted Fock space states of these free fields obey conventional statistics. We support our claims with a detailed analysis of the modifications induced in the presence of background magnetic fields, which induces additional twists by magnetic translation operators and alters the effective noncommutative geometry seen by the twisted quantum fields. When two such field theories are dual to one another, we demonstrate that only our braided physical states are covariant under the duality.

Keywords: Space-Time Symmetries, Non-Commutative Geometry.

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1. Introduction

Twisted quantum field theory is a modification of the traditional approach to noncommutative field theory [19, 42] aimed at restoring the symmetries of spacetime which are broken by noncommutativity. It was originally proposed in [15] as a means for obtaining a canonical action of the Poincaré group for quantum field theories on Moyal spaces, thus promoting twisted Poincaré invariance to a similar level as ordinary Poincaré symmetry in conventional relativistic quantum field theory. This twisted symmetry was subsequently generalized to a deformation of the bialgebra of diffeomorphisms for Moyal spaces and used to systematically construct noncommutative theories of gravity [5]. Since these foundational works a number of attempts have been made to understand the origins and implications of the twisted symmetries, and to investigate if they cure some of the complications which arise in noncommutative quantum field theory such as UV/IR mixing, violations of causality, and non-unitarity. This has led to a number of debates and different proposals for how to implement twisted Poincaré covariance on the Green's functions of the quantum field theory. One of the goals of this paper is an attempt to settle some of these issues by defining and analysing the quantization of twisted fields in as systematic a way as possible.

One of the main issues surrounding twisted quantum field theory concerns the proper definition of the correlation functions, for instance whether one should take star products when defining quantum averages of fields at separated spacetime points. Through some approaches it has been claimed that the S-matrix and Green's functions of twisted quantum field theory are perturbatively equivalent to their commutative counterparts (see e.g. [6, 22, 29, 36]). In the following we will provide an alternative interpretation of this somewhat surprising conclusion. We will show that twisted noncommutative quantum fields can be reformulated in terms of commutative quantum fields whose arguments are the coordinates of nonlocal dipole degrees of freedom. This point of view is not essentially new and has been effectively employed in e.g. [6, 7]. The novelty of our approach is that we systematically use the dipole fields to define and interpret the quantum field theory. We will see that all quantum correlation functions of twisted noncommutative fields defined using star produts can be canonically expressed in terms of those of the commutative dipole field operators. This restores causality and explains the observations above, including why the twisted quantum field theory is free from UV/IR mixing [9]. However, these averages involve nonlocal field redefinitions and the mapping back to observables in the original spacetime coordinates yields correlators which differ from that of the undeformed field theory. The intractibility of this map has been noticed in completely different contexts in [22, 25, 45].

A general framework which emcompasses quantum field theories with Hopf algebra symmetries is provided by braided quantum field theory [35] which uses combinatorial techniques of braided categories for computing quantum averages. It enables the formulation of symmetry relations among correlation functions such as Ward-Takahashi identities which can be systematically formulated on Moyal spaces [37, 38]. We will show that much of this algebraic machinery can be avoided by exploiting the formulation in terms of commutative dipole operators. Their causality property enables us to derive explicitly the functional integral defining the quantum field theory which is manifestly invariant under the twisted spacetime symmetries. As it is formulated in terms of the nonlocal dipole coordinates, the functional integration measure differs from the usual commutative one that is used in the traditional perturbative approaches to noncommutative quantum field theory which suffer from UV/IR mixing, acausality and non-unitarity. This point of view suggests that twisted quantum fields capture the infrared dynamics of noncommutative field theory which is dual to the ultraviolet dynamics of the elementary noncommutative quantum fields.

Another central issue surrounds the physical interpretation of states in the twisted Fock space for free fields. The debate does not seem to involve the deformation of the canonical commutation relations of creation and annihilation operators, which has been recently confirmed to arise from a noncommutative correspondence principle based on consistent twistings of the bilinear maps associated to Poisson and commutator brackets

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in [3]. Indeed, our definition of twisted quantum fields also leads to the same deformation. The main complication is the claim that the statistics of particle states change due to the twisting (see e.g. [9]). In this paper we will argue that no such change occurs and twisted Poincaré invariance is compatible with conventional Bose or Fermi statistics. This follows immediately from the dipole formulation which requires defining multiparticle states using braided tensor algebra, as in [22], which is compatible with the star product and the deformed coproduct of the twisted Hopf symmetry algebra. We show explicitly that the twisted states obey ordinary statistics under the twisted action of the permutation group on the Fock space.

While our argument for conventional statistics is rooted in a deep algebraic fact, the braiding and twist in the case of Moyal spaces is so simple that it simply amounts to consistent choices of sign conventions in momentum dependent phases of physical states. To further elucidate the validity of these choices, we show that only these definitions map consistently in the expected way between two noncommutative field theories which are dual to one another. This situation occurs, for instance, when the field theory involves charged fields coupled to a constant background magnetic field F. If we allow the two-form F to be a freely varying parameter, then one obtains a continuous family of twisted quantum field theories labelled by F and related to each other by duality transformations. We show that the effective noncommutative geometry seen by the twisted quantum fields is modified in this instance. In particular, there is a "self-dual" point where only a commutative description of twisted fields is available (in half the spacetime dimension). When the field theory is defined on a noncommutative torus, the flux F is quantized and there is an infinite discrete family of twisted quantum field theories related to each other by Morita equivalence. We adapt the standard duality mappings between adjoint fields to our definitions of twisted quantum fields and show that the twisted Fock space states, with our convention using braiding, transform covariantly under the Morita duality. This shows that the twisted particle states with conventional statistics are the only ones which obey the expected physical equivalence in the mapping between the dual quantum field theories. As a byproduct of our construction, we show how twisted oscillators in background fields are modified by further twists involving magnetic translation operators and relate our construction to properties of the renormalizable, duality-covariant noncommutative quantum field theories.

The outline of the remainder of this paper is as follows. Throughout this paper we will be mostly concerned with the systematic construction and analysis of twisted scalar quantum fields themselves, and not the explicit implementation of the twisted spacetime symmetries. We shall also frequently compare and contrast our results with the existing literature for clarity. In section 2 we detail the definition of twisted quantum fields in terms of dipole operators and describe their main properties. In section 3 we construct the functional integral formulation of the twisted quantum field theory and briefly study the nonlocal field redefinitions required to map correlation functions in terms of the local spacetime coordinates. In section 4 we study the twisted Fock space, and in section 5 we describe the modifications to twisted quantum field theory for charged scalar fields coupled to magnetic fields. In section 6 we derive the covariant transformation law for twisted states on a two-dimensional rational noncommutative torus, while in section 7 the construction is extended to higher dimensions and irrational noncommutativity parameters.

2. Twisted quantum fields on Moyal spaces

In this section we will define twisted quantum field theory in a way which naturally explains how the usual pathologies of noncommutative field theory, such as UV/IR mixing and nonunitarity in Minkowski space, are cured by, for example, the elimination of non-planar diagrams from the perturbation expansion. This elucidates some previous observations in the literature [9]. However, we will also stress the point that correlation functions of the field theory are *not* the same as those of ordinary commutative field theory, contrary to some previous claims [1, 6, 22, 29, 44].

2.1 Dipole coordinates

The simplest noncommutative space is the Moyal space of dimension d which is described by the associative $*$-algebra $\mathcal{A}_\theta = \mathcal{A}_\theta(\mathbb{R}^d)$ generated by hermitean coordinate operators $\hat{x}_i$ with $i = 1, \dots, d$ subject to a set of Heisenberg commutation relations. We extend this algebra by linear derivations $\hat{p}^i = -\,\mathrm{i}\,\hat{\partial}^i \in \mathrm{Aut}(\mathcal{A}_\theta)$, $i = 1, \dots, d$ to a deformed algebra of differential operators $\mathcal{D}_\theta = \mathcal{D}_\theta(\mathbb{R}^d)$. The generators of $\mathcal{D}_\theta$ obey the commutation relations

$$\begin{aligned} \big[\hat{x}_i\,,\,\hat{x}_j\big] &= \mathrm{i}\,\theta_{ij}\,, \\ \big[\hat{x}_i\,,\,\hat{p}^j\big] &= \mathrm{i}\,\delta_i{}^j\,, \\ \big[\hat{p}^i\,,\,\hat{p}^j\big] &= 0\,, \end{aligned} \tag{2.1}$$

where $\theta = (\theta_{ij})$ is a constant, positive antisymmetric $d \times d$ noncommutativity parameter matrix (so that the spacetime dimension d is even). In this paper we will work for the most part on the trivial rank one module $\mathcal{D}_\theta$ over this algebra, acting on itself by left multiplication.

There is an algebra morphism

$$\begin{aligned} \mathcal{D}_\theta &\longrightarrow \mathcal{D}_0 \\ \big(\hat{x}_i\,,\,\hat{p}^j\big) &\longmapsto \big(X_i\,,\,P^j\big) \end{aligned} \tag{2.2}$$

defined by the mapping of hermitean operators

$$\begin{aligned} X_i &\doteq \hat{x}_i + \tfrac{1}{2}\,\theta_{ij}\,\hat{p}^j\,, \\ P^i &\doteq \hat{p}^i\,. \end{aligned} \tag{2.3}$$

The algebra $\mathcal{D}_0$ generated by $\{X_i, P^j\}$ is the standard canonical commutation relation algebra

$$\begin{aligned} \big[X_i\,,\,X_j\big] &= 0\,, \\ \big[X_i\,,\,P^j\big] &= \mathrm{i}\,\delta_i{}^j\,, \\ \big[P^i\,,\,P^j\big] &= 0\,. \end{aligned} \tag{2.4}$$

Using (2.1) and the fact that the momentum operators $\hat{p}^i$ are derivatives, the new coordinate operators X_i can be rewritten in the form

$$X_i \;=\; \hat{x}_i - \tfrac{\mathrm{i}}{2}\,\theta_{ij}\,\big[\hat{\partial}^j\,,\,-\big] \;=\; \hat{x}_i - \tfrac{1}{2}\,\mathrm{ad}_{\hat{x}_i} \;=\; \tfrac{1}{2}\,\Big(\hat{x}_i + \hat{x}_i^{(R)}\Big)\;, \tag{2.5}$$

where $\hat{x}_i^{(R)}$ denotes the right action of the coordinate operators on the algebra of observables $\mathcal{A}_\theta$. The latter form of X_i can be easily seen to correspond to commuting coordinate operators [7, 8].

The operators X_i have appeared before in many different contexts. In quantum mechanics the morphism (2.2) is called the Bopp shift [12]. In the context of nonlinear integrable systems its analog is called a dressing transformation [20, 24, 46]. In this paper we will call X_i "dipole coordinates", as they are commuting but nonlocal position operators which grow with increasing centre of mass momentum transverse to their extension, due to their dipole moment $\ell_i = \frac{1}{2}\,\theta_{ij}\,p^j$. They may be thought of as parametrizing the fundamental physical excitations responsible for the nonlocal interactions of noncommutative field theory [19, 42]. Indeed, the twisted quantum field theory defined below mimicks the definitions of noncommutative dipole field theories [11, 16]. The infrared dynamics of these dipoles are dual to the ultraviolet dynamics of the elementary noncommutative quantum fields. The twisted quantum field theory that we study in this paper isolates this low-energy sector of noncommutative quantum field theory.

Let us recall the geometrical meaning of the morphism (2.2) [5, 7, 8, 37, 43]. Let

$$\xi(x) = \xi_i(x)\;\partial^i \tag{2.6}$$

be a vector field acting in the scalar representation of the diffeomorphism group $\mathrm{Diff}(\mathbb{R}^d)$. We identify the coordinates $x = (x_i)$ of $\mathbb{R}^d$ as the simultaneous eigenvalues of the operators X_i and P^i with the differential operators $-\,\mathrm{i}\,\partial^i = -\,\mathrm{i}\,\partial/\partial x_i$. A twisted diffeomorphism in $\mathcal{D}_\theta$ can then be obtained from a standard diffeomorphism generated by $\xi(x)$ by simply identifying the eigenvalues $x \in \mathbb{R}^d$ with the commutative dipole operators X themselves to obtain

$$\begin{aligned}\xi^\theta(\hat{x}) \doteq \xi_i(X)\;\partial^i \;&=\; \xi_i\big(\hat{x} - \tfrac{\mathrm{i}}{2}\,\theta\cdot\hat{\partial}\,\big)\;\hat{\partial}^i \\ &= \xi_i(\hat{x})\;\hat{\partial}^i + \sum_{n=1}^\infty\,\Big(-\frac{\mathrm{i}}{2}\Big)^n\;\frac{1}{n!}\,\theta_{i_1j_1}\cdots\theta_{i_nj_n}\,\big(\hat{\partial}^{i_1}\cdots\hat{\partial}^{i_n}\xi_i(\hat{x})\big)\;\hat{\partial}^{j_1}\cdots\hat{\partial}^{j_n}\;\hat{\partial}^i\;.\end{aligned} \tag{2.7}$$

This formula agrees with the standard expression [5, 43] for the action of twisted diffeomorphisms on $\mathcal{A}_\theta$ as $(\xi^\theta\triangleright f)(\hat{x}) \doteq \xi^\theta(\hat{x})\,f(\hat{x}) = \xi(f)$.

This definition ensures that the symmetry generators ξ^θ act covariantly on the algebra $\mathcal{A}_\theta$. For any pair of functions $f, g \in \mathcal{A}_0$ one has the twisted Leibniz rule

$$\xi^\theta\triangleright(f\star g) = \mu_\theta\big(\Delta_\theta(\xi^\theta)\triangleright(f\otimes g)\big)\;, \tag{2.8}$$

where

$$f\star g \;=\; \mu_\theta(f\otimes g) \;=\; \big(\mathcal{F}^{(1)}\triangleright f\big)\,\big(\mathcal{F}^{(2)}\triangleright g\big) \tag{2.9}$$

is the noncommutative star product on $\mathcal{A}_0$ with

$$\mathcal{F} \;=\; \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \;=\; \exp\big(-\tfrac{\mathrm{i}}{2}\,\theta_{ij}\,P^i \otimes P^j\big) \tag{2.10}$$

an abelian Drinfeld twist associated to the algebra $\mathcal{D}_\theta$. Here we have used a Sweedler notation with a suppressed summation index, and the twisted coproduct is given by

$$\Delta_\theta\big(\xi^\theta\big) = \mathcal{F}^{-1}\left(\xi^\theta \otimes \mathrm{id}_{\mathcal{A}_\theta} + \mathrm{id}_{\mathcal{A}_\theta} \otimes \xi^\theta\right)\mathcal{F} \ . \tag{2.11}$$

The usual observables of noncommutative field theory are defined by substituting the operators $\hat{x}$ for the arguments x of functional expressions for commutative observables $f \in \mathcal{A}_0$ [19, 42]. With a suitable ordering this defines the Weyl transform $W : \mathcal{A}_0 \to \mathcal{A}_\theta$. In what follows we will be interested in the subalgebra of $\mathcal{A}_\theta$ obtained by substituting instead the dipole coordinates X, according to prescription stated above. These are known as twisted fields and are the ones which transform covariantly with respect to the action of the deformed vector fields (2.7) (when expressed in dipole coordinates). Since the dipole operators are (linear) combinations of the coordinates and momenta, they yield nonlocal field redefinitions containing an infinite number of derivatives in terms of the original variables x. Thus even though the new coordinates X_i commute, their insertion leads to an inherent nonlocality characteristic of noncommutative field theory.

2.2 Free twisted quantum fields

Consider a free real scalar quantum field $\check{\phi}(x)$ of mass $m > 0$ in second quantization on ordinary Minkowski spacetime in d dimensions. We denote the simultaneous eigenvalues of the momentum operators P^i by $p = (p^i) = (p^0, \boldsymbol{p})$, where $\boldsymbol{p} \in \mathbb{R}^{d-1}$ and the (upper) mass-shell relation gives

$$p^0 = \sqrt{m^2 + \boldsymbol{p}^2} \ . \tag{2.12}$$

The mode expansion of the relativistic field is given by

$$\check{\phi}(x) = \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}}{2p^0}\ \Big(\check{a}(p)\ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot x} + \check{a}^\dagger(p)\ \mathrm{e}^{\,\mathrm{i}\,p\cdot x}\Big) \ , \tag{2.13}$$

which realizes it as an operator-valued tempered distribution in terms of the standard representation of the canonical commutation relation algebra on the bosonic Fock space $\mathcal{H}$. The creation and annihilation operators obey

$$\begin{aligned} \check{a}(p)\,\check{a}(q) &= \check{a}(q)\,\check{a}(p)\,, \\ \check{a}^\dagger(p)\,\check{a}^\dagger(q) &= \check{a}^\dagger(q)\,\check{a}^\dagger(p)\,, \\ \check{a}(p)\,\check{a}^\dagger(q) &= \check{a}^\dagger(q)\,\check{a}(p) + 2p^0\ \delta^d(p-q) \ . \end{aligned} \tag{2.14}$$

Let us now substitute the commuting dipole coordinate operators X in place of their eigenvalues x. To make this *substitution* unambiguous, we define it in terms of the symmetrically ordered parity operator ${}^{\circ}_{\circ}\,\delta^d(\hat{A}\,)\,{}^{\circ}_{\circ} \doteq \int \frac{\mathrm{d}^d p}{(2\pi)^d}\ \exp\left(\mathrm{i}\,p\cdot\hat{A}\,\right)$, so that for example the Weyl transform $W:\mathcal{A}_0\to\mathcal{A}_\theta$ of a function $f\in\mathcal{A}_0$ is given by

$$
\begin{aligned}
W\big[f(x)\big] \;&=\; f(\hat{x}) \doteq \int \mathrm{d}^d\xi \; f(\xi) \; {}^{\circ}_{\circ}\,\delta^d(\xi-\hat{x})\,{}^{\circ}_{\circ} \\
&= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int \mathrm{d}^d\xi \; f(\xi) \;\exp\big(\mathrm{i}\,p\cdot(\xi-\hat{x})\big) \ . \qquad (2.15)
\end{aligned}
$$

This definition leads directly to the Moyal product

$$
f(\hat{x})\,g(\hat{x}) = (f\star g)(\hat{x}) \ . \qquad (2.16)
$$

For the positive energy component of the quantum field $\check{\phi}$ at momentum p, we then have

$$
\begin{aligned}
\check{\phi}^{+}_{p}(X) &= \check{\phi}^{+}_{p}\big(\hat{x}+\tfrac12\,\theta\cdot\hat{p}\big) \\
&= \int \mathrm{d}^d\xi \;\; \mathrm{e}^{\,-\mathrm{i}\,p\cdot\xi}\;\check{a}(p)\;{}^{\circ}_{\circ}\,\delta^d\big(\xi-\hat{x}-\tfrac12\,\theta\cdot\hat{p}\big)\,{}^{\circ}_{\circ} \;=\; \check{a}(p)\;\;\mathrm{e}^{\,-\frac{\mathrm{i}}{2}\,p\cdot\theta\cdot\hat{p}}\;\;\mathrm{e}^{\,-\mathrm{i}\,p\cdot\hat{x}} \ . \qquad (2.17)
\end{aligned}
$$

This realizes the twisted quantum field $\check{\phi}(X)$ as an operator on $\mathcal{D}_\theta\otimes\mathcal{H}$.

Alternatively, we can represent $\check{\phi}(X)$ as an operator on $\mathcal{A}_\theta\otimes\mathcal{H}$ via the Weyl transform by defining a set of *twisted oscillators* (see for example [3, 9, 22, 25]), i.e. by defining a new set of creation and annihilation operators $\check{a}_\theta(p),\check{a}^\dagger_\theta(p)$ in terms of the original ones. eq. (2.17) suggests the definition

$$
\begin{aligned}
\check{a}_\theta(p) &\doteq \check{a}(p)\;\;\mathrm{e}^{\,-\frac{\mathrm{i}}{2}\,p\cdot\theta\cdot\check{P}} \ , \\
\check{a}^\dagger_\theta(p) &\doteq \check{a}^\dagger(p)\;\;\mathrm{e}^{\,\frac{\mathrm{i}}{2}\,p\cdot\theta\cdot\check{P}}
\end{aligned}
\qquad (2.18)
$$

where

$$
\check{P}^i \doteq \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}}{2p^0}\;p^i\;\check{a}^\dagger(p)\,\check{a}(p) \qquad (2.19)
$$

is the momentum operator on the Fock space $\mathcal{H}$. It can be easily shown that the ordering of the oscillators $\check{a}(p)$ and exponentials in (2.18) is immaterial, and that in the definition of the momentum operator (2.19) we could as well have used the $\check{a}_\theta(p)$ oscillators instead, due to antisymmetry of the tensor θ_{ij}.

The new oscillators generate the twisted canonical commutation relation algebra

$$
\begin{aligned}
\check{a}_\theta(p)\,\check{a}_\theta(q) &= \check{a}_\theta(q)\,\check{a}_\theta(p)\;\;\mathrm{e}^{\,\mathrm{i}\,p\cdot\theta\cdot q} \ , \\
\check{a}^\dagger_\theta(p)\,\check{a}^\dagger_\theta(q) &= \check{a}^\dagger_\theta(q)\,\check{a}^\dagger_\theta(p)\;\;\mathrm{e}^{\,\mathrm{i}\,p\cdot\theta\cdot q} \ , \\
\check{a}_\theta(p)\,\check{a}^\dagger_\theta(q) &= \check{a}^\dagger_\theta(q)\,\check{a}_\theta(p)\;\;\mathrm{e}^{\,-\mathrm{i}\,p\cdot\theta\cdot q}+2p^0\;\delta^d(p-q) \ .
\end{aligned}
\qquad (2.20)
$$

The field operators corresponding to these twisted oscillators are given by

$$
\check{\phi}^\theta(x) = \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}}{2p^0}\;\Big(\check{a}_\theta(p)\;\;\mathrm{e}^{\,-\mathrm{i}\,p\cdot x}+\check{a}^\dagger_\theta(p)\;\;\mathrm{e}^{\,\mathrm{i}\,p\cdot x}\Big) \ . \qquad (2.21)
$$

They can be used to canonically generate the twisted quantum fields above through their Weyl transform

$$
\check{\phi}(X) = \check{\phi}^\theta(\hat{x}) \ . \qquad (2.22)
$$

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Using the twisted commutation relations (2.20), we can compute the star commutator of the scalar field (2.21) at separated points. One has

$$
\begin{aligned}
&\big[\check\phi^\theta(x) \stackrel{\star}{,} \check\phi^\theta(y)\big] \\
&\quad = \int \frac{\mathrm{d}^{d-1}\boldsymbol{p}}{2p^0} \int \frac{\mathrm{d}^{d-1}\boldsymbol{q}}{2q^0} \Big[\Big(\check a_\theta(p)\check a_\theta(q)\,\mathrm{e}^{\,-\mathrm{i}\,p\cdot x}\star \mathrm{e}^{\,-\mathrm{i}\,q\cdot y} - \check a_\theta(q)\check a_\theta(p)\,\mathrm{e}^{\,-\mathrm{i}\,q\cdot y}\star \mathrm{e}^{\,-\mathrm{i}\,p\cdot x}\Big) \\
&\qquad\qquad + \Big(\check a_\theta(p)\check a_\theta^\dagger(q)\,\mathrm{e}^{\,-\mathrm{i}\,p\cdot x}\star \mathrm{e}^{\,\mathrm{i}\,q\cdot y} - \check a_\theta^\dagger(q)\check a_\theta(p)\,\mathrm{e}^{\,\mathrm{i}\,q\cdot y}\star \mathrm{e}^{\,-\mathrm{i}\,p\cdot x}\Big) \\
&\qquad\qquad + \Big(\check a_\theta^\dagger(p)\check a_\theta(q)\,\mathrm{e}^{\,\mathrm{i}\,p\cdot x}\star \mathrm{e}^{\,-\mathrm{i}\,q\cdot y} - \check a_\theta(q)\check a_\theta^\dagger(p)\,\mathrm{e}^{\,-\mathrm{i}\,q\cdot y}\star \mathrm{e}^{\,\mathrm{i}\,p\cdot x}\Big) \\
&\qquad\qquad + \Big(\check a_\theta^\dagger(p)\check a_\theta^\dagger(q)\,\mathrm{e}^{\,\mathrm{i}\,p\cdot x}\star \mathrm{e}^{\,\mathrm{i}\,q\cdot y} - \check a_\theta^\dagger(q)\check a_\theta^\dagger(p)\,\mathrm{e}^{\,\mathrm{i}\,q\cdot y}\star \mathrm{e}^{\,\mathrm{i}\,p\cdot x}\Big)\Big] \ .
\end{aligned}
\tag{2.23}
$$

From (2.1), (2.16) and the Baker-Campbell-Hausdorff formula one has the identity

$$
\mathrm{e}^{\,\mathrm{i}\,p\cdot x}\star \mathrm{e}^{\,\mathrm{i}\,q\cdot y} = \mathrm{e}^{\,\mathrm{i}\,p\cdot x+\mathrm{i}\,q\cdot y}~\mathrm{e}^{\,-\frac{\mathrm{i}}{2}\,p\cdot\theta\cdot q} \ . \tag{2.24}
$$

It follows that every bracketed term in (2.23) vanishes but one, leaving the result

$$
\big[\check\phi^\theta(x) \stackrel{\star}{,} \check\phi^\theta(y)\big] = \mathrm{i}\,\int \frac{\mathrm{d}^dp}{p^0}~\sin p\cdot(x-y)~\delta\Big(p^0-\sqrt{m^2+\boldsymbol{p}^2}~\Big) \ . \tag{2.25}
$$

The integral vanishes for spacelike separated points x, y, and hence the algebra of twisted real scalar fields with the star product is microcausal, just as in the commutative case. This property is consistent with their identification (2.22) in terms of commutative dipole fields and was also observed in [22].

However, as the dipole coordinate X is defined in terms of *both* coordinates and momenta, the field $\check\phi(X)$ is expressed in a nonlocal way in terms of the original position variable x. It follows that local observables written in terms of $\check\phi(X)$ correspond to nonlocal observables in terms of the fields $\check\phi(x)$ containing an infinite number of derivatives. This is easily seen by noting that the Green's functions of the quantum field theory are built from the vacuum expectation values of products of the fields

$$
\begin{aligned}
\check\phi(X^1)\cdots\check\phi(X^n) &= \check\phi^\theta(\hat x^1)\cdots\check\phi^\theta(\hat x^n) ~=~ W\big[\check\phi^\theta(x^1)\star\cdots\star\check\phi^\theta(x^n)\big] \\
&= W\Big[\prod_{a<b}\exp\Big(-\frac{\mathrm{i}}{2}\,\frac{\partial}{\partial x_i^a}\,\theta_{ij}\,\frac{\partial}{\partial x_j^b}\Big)\,\check\phi^\theta(x^1)\cdots\check\phi^\theta(x^n)\Big] \ .
\end{aligned}
\tag{2.26}
$$

In turn, this implies that there are no non-planar diagrams in the perturbation expansion of any interacting model built on these fields, and hence the usual undesirable features of noncommutative quantum field theory such as UV/IR mixing and non-unitarity in Minkowski signature do not show up in twisted quantum field theory. There is, however, a nonlocal correspondence between the correlation functions of the twisted theory and those of the original untwisted noncommutative field theory. This renders an alternative perspective on the problem of the computability of twisted quantum field theory described in [45]. We describe some aspects of this correspondence in the next section.

3. Functional formulation of twisted quantum field theory

The definition of the functional integral which defines a quantum field theory invariant under twisted spacetime symmetries can be given formally in the context of braided quantum field theory [35, 36, 38] (see also [27]). In this section we will give a more pragmatic definition which avoids the use of this abstract algebraic machinery by exploiting the formulation in terms of dipole fields. This quantization of the twisted field theory makes it qualitatively similar to noncommutative dipole field theories [16]. The mapping of correlators in the dipole coordinates back to correlators in the original spacetime coordinates can be presumably achieved using the braided formalism, though we will not pursue this issue here.

3.1 Twist invariant functional integral

To quantize the noncommutative scalar field theory defined in terms of twisted fields using functional methods, we need to construct a measure for the path integral which defines the twisted quantum field theory with the manifest twisted spacetime symmetries. This can be done by using the formalism of dipole coordinates and mimicking the standard treatment of the functional integral in the commutative case. Notice first of all that the field theory with twisted fields and the star product can be canonically quantized, since the equal time canonical commutation relations hold. Writing $x=(\boldsymbol{x};t)$ with $\boldsymbol{x}\in\mathbb{R}^{d-1}$, in the free field case this follows by applying time derivatives to (2.25).

The equal time canonical commutation relation algebra is

$$\begin{aligned}\big[\check\phi^\theta(\boldsymbol{x};t)\overset{\star}{,}\check\phi^\theta(\boldsymbol{y};t)\big] &= 0\,,\\ \big[\check\Pi^\theta(\boldsymbol{x};t)\overset{\star}{,}\check\Pi^\theta(\boldsymbol{y};t)\big] &= 0\,,\\ \big[\check\phi^\theta(\boldsymbol{x};t)\overset{\star}{,}\check\Pi^\theta(\boldsymbol{y};t)\big] &= \mathrm{i}\,\delta^{d-1}(\boldsymbol{x}-\boldsymbol{y})\end{aligned}\tag{3.1}$$

where $\check\Pi^\theta(\boldsymbol{x};t)$ is the momentum operator conjugate to the quantum field $\check\phi^\theta(\boldsymbol{x};t)$. One can therefore apply all the usual machinery of coherent states $|\phi;t\rangle,|\Pi;t\rangle$ defined as the simultaneous eigenstate functionals of the quantum field operators with

$$\check\phi(\boldsymbol{X};t)|\phi;t\rangle \doteq \phi(\boldsymbol{\xi};t)\,|\phi;t\rangle \qquad\text{and}\qquad \check\Pi(\boldsymbol{X};t)|\Pi;t\rangle \doteq \Pi(\boldsymbol{\xi};t)\,|\Pi;t\rangle\tag{3.2}$$

where $\boldsymbol{\xi}\in\mathbb{R}^{d-1}$ are the eigenvalues of the spatial dipole operators $\boldsymbol{X}$. They span a functional Hilbert space $\underline{\mathcal{H}}$ and have the usual inner products

$$\begin{aligned}\langle\phi_1;t|\phi_2;t\rangle &= \delta[\phi_1-\phi_2]\,,\\ \langle\Pi_1;t|\Pi_2;t\rangle &= \delta[\Pi_1-\Pi_2]\,,\\ \langle\phi;t|\Pi;t\rangle &= \frac{1}{\sqrt{2\pi}}\,\exp\left(\mathrm{i}\int\mathrm{d}^{d-1}\boldsymbol{\xi}\;\Pi(\boldsymbol{\xi};t)\,\phi(\boldsymbol{\xi};t)\right)\,.\end{aligned}\tag{3.3}$$

We only have to keep track of star products when they appear.

Let us start with the propagator

$$\langle\,\underline{\Psi}_{\,\rm out}(t_{\rm f})|\,\underline{\Psi}_{\,\rm in}(t_{\rm i})\rangle\tag{3.4}$$

describing the S-matrix element between in and out states. We partition the time interval $[t_{\rm i}, t_{\rm f}]$ into intermediate times $t_{\rm i} = t_0 < t_1 < \cdots < t_N = t_{\rm f}$, and at the end take the limit $N \to \infty$. We use the completeness relation

$$\mathrm{id}_{\underline{\mathcal{H}}} = \int \prod_{\boldsymbol{\xi}\in\mathbb{R}^{d-1}} \mathrm{d}\phi(\boldsymbol{\xi};t)\ |\phi;t\rangle\langle\phi;t| \tag{3.5}$$

with a redundant notation to single out the time dependence. Inserting the identity (3.5) for each time t_l into the propagator (3.4), we obtain

$$\begin{aligned}\langle\,\underline{\Psi}_{\,\rm out}(t_{\rm f})|\,\underline{\Psi}_{\,\rm in}(t_{\rm i})\rangle \ &= \ \lim_{N\to\infty}\prod_{l=0}^{N}\int\prod_{\boldsymbol{\xi}_l\in\mathbb{R}^{d-1}} \mathrm{d}\phi_l(\boldsymbol{\xi}_l;t_l)\langle\,\underline{\Psi}_{\rm out}(t_N)|\phi_N;t_N\rangle\langle\phi_N;t_N|\phi_{N-1};t_{N-1}\rangle \\ &\qquad\times\langle\phi_{N-1};t_{N-1}|\phi_{N-2};t_{N-2}\rangle\cdots\langle\phi_1;t_1|\phi_0;t_0\rangle\,\langle\phi_0;t_0|\,\underline{\Psi}_{\,\rm in}(t_0)\rangle.\end{aligned} \tag{3.6}$$

Now we use the fact that the quantum hamiltonian $H[\check{\Pi},\check{\phi}\,]$ of the scalar field theory generates time translations with

$$\langle\phi_2;t_2|\phi_1;t_1\rangle = \langle\phi_2;t_1|\,\mathrm{e}^{\,-\,\mathrm{i}\,(t_2-t_1)\,H[\check{\Pi},\check{\phi}]}|\phi_1;t_1\rangle\ . \tag{3.7}$$

Inserting the completeness relation

$$\mathrm{id}_{\underline{\mathcal{H}}} = \int \prod_{\boldsymbol{\xi}\in\mathbb{R}^{d-1}} \mathrm{d}\Pi(\boldsymbol{\xi};t)\ |\Pi;t\rangle\langle\Pi;t| \tag{3.8}$$

at times $t_0, t_1, \ldots, t_{N-1}$ into (3.6) and using the inner products (3.3), we obtain the standard expression for the phase space path integral

$$\begin{aligned}&\langle\,\underline{\Psi}_{\,\rm out}(t_{\rm f})|\,\underline{\Psi}_{\,\rm in}(t_{\rm i})\rangle \\ &\quad= \lim_{N\to\infty}\int\prod_{l=0}^{N}\prod_{\boldsymbol{\xi}_l\in\mathbb{R}^{d-1}}\mathrm{d}\phi_l(\boldsymbol{\xi}_l;t_l)\prod_{l=0}^{N-1}\prod_{\boldsymbol{\xi}_l\in\mathbb{R}^{d-1}}\frac{\mathrm{d}\Pi_l(\boldsymbol{\xi}_l;t_l)}{\sqrt{2\pi}}\ \langle\,\underline{\Psi}_{\,\rm out}(t_N)|\phi_N;t_N\rangle \\ &\qquad\qquad\times\exp\left(\mathrm{i}\sum_{l=0}^{N-1}\big(\Pi_l\,(\phi_{l+1}-\phi_l)-\delta t_l\ H[\Pi_l,\phi_l]\big)\right)\langle\phi_0;t_0|\,\underline{\Psi}_{\,\rm in}(t_0)\rangle\end{aligned} \tag{3.9}$$

with $\delta t_l = t_{l+1} - t_l$. The quantum hamiltonian $H[\check{\Pi},\check{\phi}\,]$ is the original one defined in terms of the noncommutative star product, and hence the exponential function in (3.9) reproduces the usual Boltzmann weight with the action functional of the untwisted noncommutative field theory.

The measure in (3.9) is a product of the Lebesgue measures $\mathrm{d}\phi(\boldsymbol{\xi};t)\ \mathrm{d}\Pi(\boldsymbol{\xi};t)$ over the eigenvalues of the dipole operators $\boldsymbol{X}$. This defines the twist invariant functional integration measure. By means of the identification (2.22), we see that the hamiltonian written in terms of the field variables $\phi(X)$ is the hamiltonian of the underlying commutative field theory. However, our S-matrix elements are not commutative ones but appear to agree with those computed in [13]. Moreover, our functional integration measure is different from the naive commutative one used in e.g. [44].

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3.2 Coherent state representation of dipole operators

We will now elucidate the correspondence between the set of noncommuting coordinates $\hat{x}$ and the commutative dipole coordinates X, required to match correlation functions defined by the functional integral (3.9). For this, we rewrite the standard formalism of coherent state representations appropriate to the field dependence of twisted quantum field theory. For clarity we restrict the construction to $d = 2$ spacetime dimensions. First, we pass to complex coordinate operators

$$\begin{aligned} \hat{z} &\doteq \hat{x}_1 + \mathrm{i}\,\hat{x}_2\,, \\ \hat{z}^\dagger &\doteq \hat{x}_1 - \mathrm{i}\,\hat{x}_2 \end{aligned} \tag{3.10}$$

obeying the commutation relations

$$\left[\hat{z}\,,\,\hat{z}^\dagger\right] \;=\; 2\theta_{12} \;\doteq\; 2\theta\;. \tag{3.11}$$

We also define corresponding complex derivations $\hat{\partial}, \hat{\bar{\partial}}$ in such a way that they fulfill the commutation relations

$$\begin{aligned} \left[\hat{\partial}\,,\,\hat{z}\right] &= 1 \;=\; \left[\hat{\bar{\partial}}\,,\,\hat{z}^\dagger\right], \\ \left[\hat{\partial}\,,\,\hat{z}^\dagger\right] &= 0 \;=\; \left[\hat{\bar{\partial}}\,,\,\hat{z}\right], \\ \left[\hat{\partial}\,,\,\hat{\bar{\partial}}\right] &= 0\;. \end{aligned} \tag{3.12}$$

We can then introduce a module over the algebra $\mathcal{A}_\theta$ spanned by coherent states which are the normalized eigenstates of the coordinate operators $\hat{z}$ given by

$$|\zeta\rangle \;=\; \exp\left(-\frac{|\zeta|^2}{4\theta}\right)\,\exp\left(\frac{\zeta\,\hat{z}^\dagger}{2\theta}\right)|\zeta=0\rangle \qquad \text{with} \quad \hat{z}|\zeta\rangle \;=\; \zeta\,|\zeta\rangle\,, \tag{3.13}$$

where $\zeta \in \mathbb{C}$.

We now define the commutative complex dipole coordinates $Z, Z^\dagger$ by

$$\begin{aligned} Z &\doteq X_1 + \mathrm{i}\,X_2 \;=\; \hat{z} - \theta\,\hat{\bar{\partial}}\,, \\ Z^\dagger &\doteq X_1 - \mathrm{i}\,X_2 \;=\; \hat{z}^\dagger + \theta\,\hat{\partial} \end{aligned} \tag{3.14}$$

which obey

$$\left[Z\,,\,Z^\dagger\right] = 0\;. \tag{3.15}$$

In order to define the delta-function in these new coordinates, we need to compute the matrix elements of the plane wave operator

$$\begin{aligned} \langle\zeta'|\exp\left(\mathrm{i}\,p\cdot X\right)|\zeta\rangle \;&=\; \langle\zeta'|\exp\left(\tfrac{\mathrm{i}}{2}\left(p\,Z^\dagger + p^*\,Z\right)\right)|\zeta\rangle \\ &=\; \langle\zeta'|\exp\left(\tfrac{\mathrm{i}}{2}\,\theta\left(p\,\hat{\partial} - p^*\,\hat{\bar{\partial}}\right)\right)|\zeta\rangle\,\exp\left(\tfrac{\mathrm{i}}{2}\left(p\,\zeta'^* + p^*\,\zeta\right)\right)\;. \end{aligned} \tag{3.16}$$

One can straightforwardly derive the identities

$$\begin{aligned} \langle\eta|\hat{\partial}^{\,l}|\zeta\rangle \;&=\; \left(\tfrac{\eta^*}{2\theta}\right)^l\,\langle\eta|\zeta\rangle\;, \\ \langle\eta|\hat{\bar{\partial}}^{\,l}|\zeta\rangle \;&=\; \left(-\tfrac{\zeta}{2\theta}\right)^l\,\langle\eta|\zeta\rangle \end{aligned} \tag{3.17}$$

from which we find

$$\begin{aligned} \langle\zeta'|\exp(\mathrm{i}\,p\cdot X)|\zeta\rangle &= \langle\zeta'|\exp\big(\tfrac{\mathrm{i}}{2}\,\theta\,(p\,\tfrac{\zeta'^*}{2\theta}+p^*\,\tfrac{\zeta}{2\theta})\big)|\zeta\rangle\,\exp\big(\tfrac{\mathrm{i}}{2}\,(p\,\zeta'^*+p^*\,\zeta)\big) \\ &= \exp\big(\tfrac{3\,\mathrm{i}}{4}\,(p\,\zeta'^*+p^*\,\zeta)\big)\,\langle\zeta'|\zeta\rangle\ . \end{aligned} \tag{3.18}$$

We can compare the matrix element (3.18) with the analogous one for the standard noncommutative coordinates $\hat{z},\hat{\bar{z}}$. One finds

$$\begin{aligned} \langle\zeta'|\exp(\mathrm{i}\,p\cdot\hat{x})|\zeta\rangle &= \langle\zeta'|\exp\big(\tfrac{\mathrm{i}}{2}\,(p\,\hat{z}^\dagger+p^*\,\hat{z})\big)|\zeta\rangle \\ &= \exp\big(\tfrac{\mathrm{i}}{2}\,(p\,\zeta'^*+p^*\,\zeta)-\tfrac{\theta}{4}\,|p|^2\big)\,\langle\zeta'|\zeta\rangle\ . \end{aligned} \tag{3.19}$$

The difference between the two matrix elements (3.18) and (3.19) emphasizes the commutativity of the dipole coordinates X, as well as the nonlocal relationship between them and the noncommutative coordinates $\hat{x}$.

One can also easily derive the matrix elements

$$\langle\zeta'|Z|\zeta\rangle = \big(\zeta-\tfrac{1}{2}\,\zeta'^*\big)\,\langle\zeta'|\zeta\rangle \qquad \text{and} \qquad \langle\zeta'|Z^\dagger|\zeta\rangle = \big(\zeta'^*-\tfrac{1}{2}\,\zeta\big)\,\langle\zeta'|\zeta\rangle\ . \tag{3.20}$$

From these expressions one can derive a representation of the complex dipole field operators in the coherent state basis as

$$Z = \int \mathrm{d}^2\zeta\ \big(\zeta-\tfrac{1}{2}\,\zeta^*\big)\ |\zeta\rangle\langle\zeta| \qquad \text{and} \qquad Z^\dagger = \int \mathrm{d}^2\zeta\ \big(\zeta^*-\tfrac{1}{2}\,\zeta\big)\ |\zeta\rangle\langle\zeta|\ . \tag{3.21}$$

This is to be contrasted with the coherent state representation of the noncommutative complex coordinate operators, given from the definition (3.13) by

$$\hat{z} = \int \mathrm{d}^2\zeta\ \zeta\ |\zeta\rangle\langle\zeta| \qquad \text{and} \qquad \hat{z}^\dagger = \int \mathrm{d}^2\zeta\ \zeta^*\ |\zeta\rangle\langle\zeta|\ . \tag{3.22}$$

The nonanalytic dependence of the dipole coordinate operators is analogous to that in the definition of Bargmann space in the quantum mechanics of the lowest Landau Level, where the projection onto the ground state induces the representation $\zeta^*\mapsto\partial/\partial\zeta$. This allows one to deal with nonanalytic potentials in a space of analytic functions, at the expense of locality [10, 23].

4. Braided tensor algebra on the twisted Fock space

We have seen that the twisted quantum field theory, although commutative, has nontrivial features because of its nonlocality in the original coordinate variables. The perturbation expansion of any observable, however, has no nonplanar diagrams because of the twist of the quantum fields. In this section we will explore the physical meaning of the twisted oscillators $\check{a}_\theta(p)$ introduced in (2.18). We will do this by investigating the structure of the Fock space $\mathcal{H}^\theta$ built on the quantum operators $\check{a}_\theta(p)$.

4.1 Action of the symmetric group

The Fock space $\mathcal{H}$ of the untwisted quantum field theory is graded by particle number $N \in \mathbb{N}_0$ as

$$\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N \ . \tag{4.1}$$

The one-dimensional subspace $\mathcal{H}_0$ is spanned by the vacuum state $|\Omega\rangle$ annihilated by the oscillators $\check{a}(p)$. From the definition (2.18) it follows that the vector $|\Omega\rangle$ is the (unique) vacuum state of the twisted oscillators $\check{a}_\theta(p)$. Moreover, since for any momentum p the Fock space number operator can be written as

$$\check{N}_p \ = \ \check{a}^\dagger(p)\, \check{a}(p) \ = \ \check{a}^\dagger_\theta(p)\, \check{a}_\theta(p) \ , \tag{4.2}$$

it follows that any monomial in the twisted oscillators has the naive particle number. Thus $\mathcal{H}_0^\theta = \mathcal{H}_0$ and the grading operator on the twisted Fock space $\mathcal{H}^\theta$ is the same as that on (4.1). The one-particle sector $\mathcal{H}_1^\theta$ is spanned by the vectors $|p\rangle_\theta \doteq \check{a}^\dagger_\theta(p)|\Omega\rangle = \check{a}^\dagger(p)|\Omega\rangle$. Thus $\mathcal{H}_1^\theta = \mathcal{H}_1$ and hence single-particle states are momentum eigenstates which are also unaffected by the twist. This is consistent with the fact that the twisting does not change the coproducts (2.11) of the momentum operators (2.19).

Let us now consider an indecomposable two-particle state in $\mathcal{H}_2^\theta$ given by

$$|p\rangle_\theta \otimes |q\rangle_\theta \doteq \check{a}^\dagger_\theta(p)\, \check{a}^\dagger_\theta(q)|\Omega\rangle \ . \tag{4.3}$$

The permutation of the two particles is determined by a flip operator σ which preserves the grading on $\mathcal{H}^\theta$. Using the twisted commutation relations (2.20) it is given explicitly by

$$\begin{aligned} \sigma|p\rangle_\theta \otimes |q\rangle_\theta \ &\doteq \ |q\rangle_\theta \otimes |p\rangle_\theta = \check{a}^\dagger_\theta(q)\, \check{a}^\dagger_\theta(p)|\Omega\rangle \\ &= \ \mathrm{e}^{-\,\mathrm{i}\, p\cdot\theta\cdot q}\, \check{a}^\dagger_\theta(p)\, \check{a}^\dagger_\theta(q)|\Omega\rangle \ = \ \mathrm{e}^{-\,\mathrm{i}\, p\cdot\theta\cdot q}\, |p\rangle_\theta \otimes |q\rangle_\theta \ . \end{aligned} \tag{4.4}$$

It is easy to see from this equation that the flip operator σ fails to be an involution of $\mathcal{H}^\theta$. To generate a representation of the permutation group S_N on $\mathcal{H}^\theta$, we represent the Drinfeld twist (2.10) as an operator $\check{\mathcal{F}}$ on $\mathcal{H}^\theta$ by replacing the translation generators P^i by the second quantized momentum operators (2.19) acting on $\mathcal{H}^\theta$. Using this Fock space representation we define the twisted flip operator

$$\sigma_\theta \doteq \check{\mathcal{F}} \circ \sigma \circ \check{\mathcal{F}}^{-1} \ . \tag{4.5}$$

Then σ_θ is an involution of $\mathcal{H}^\theta$. The representation of the twist operator $\check{\mathcal{F}}$ on $\mathcal{H}_2^\theta$ extends to arbitrary tensor products of single-particle states where it satisfies analogous equations.

The meaning of the expression (4.5) can be understood as follows. The flip operator σ defines a representation of the permutation group on the untwisted Fock space $\mathcal{H}$. After the twist, it fails to be a representation of the symmetric group on $\mathcal{H}^\theta$, until we twist the representation itself, which restores the Yang-Baxter equations giving the relations of S_N. Applying the twisted flip operator to a two-particle state yields

$$\sigma_\theta|p\rangle_\theta \otimes |q\rangle_\theta \ = \ \mathrm{e}^{\,\mathrm{i}\, p\cdot\theta\cdot q}\, \check{a}^\dagger_\theta(q)\, \check{a}^\dagger_\theta(p)|\Omega\rangle \ = \ \check{a}^\dagger_\theta(p)\, \check{a}^\dagger_\theta(q)|\Omega\rangle \ = \ |p\rangle_\theta \otimes |q\rangle_\theta \ . \tag{4.6}$$

The relationship between the original state and its image under transposition is nonlocal in the naive way, because of the appearence of a differential operator with infinitely many derivatives.

It is interesting to observe that, whilst in the generic case the operator σ is not an involution of the twisted Fock space, by (4.4) it squares to the identity when the momenta of the particles belong to an appropriate lattice, as is the case when the field theory is defined on some torus. Consider for definiteness the two-dimensional case, and set

$$p^i \;=\; \frac{2\pi}{L_i}\, n_i \qquad \text{with} \quad n_i \in \mathbb{Z}\;, \tag{4.7}$$

where L_i, $i = 1, 2$ are the sides of a rectangular lattice $\Gamma \cong \mathbb{Z}^2$. Then the argument of the exponential in (4.4) is of the form

$$p \cdot \theta \cdot q = 2\pi\, \frac{2\pi\,\theta}{L_1\,L_2}\,(n_1\,m_2 - m_1\,n_2)\,, \tag{4.8}$$

and the exponential is unity whenever the quantity $\frac{2\pi\,\theta}{L_1\,L_2}$ is an integer. When this happens, we can give the canonical meaning to the exchange of two particles. If our quantum field theory is defined on a torus whose area is quantized according to

$$n \cdot \mathrm{Area} \;=\; 2\pi\,\theta \qquad \text{for} \quad n \in \mathbb{N}\,, \tag{4.9}$$

then there is no need to twist the flip operator as well. In this case the Fock space $\mathcal{H}^\theta$ built by means of the twisted creation and annihilation operators $\check{a}_\theta(p), \check{a}_\theta^\dagger(p)$ is bosonic. This simple observation will be elucidated and generalized in sections 6 and 7. An analogous fermionic Fock space can be constructed by starting from untwisted fermionic field operators.

In the general case, the twisted flip operator σ_θ, together with its natural extensions to higher number multiparticle states, provides a representation of the permutation group on the twisted Fock space $\mathcal{H}^\theta$. It would be interesting to relate this action of S_N to appropriate generalizations of the known braid group representations to accomodate our commutation relations, which may be expressed as

$$|p\rangle_\theta \otimes |q\rangle_\theta \;=\; \mathrm{e}^{\,\mathrm{i}\,p\cdot\theta\cdot q}\,|q\rangle_\theta \otimes |p\rangle_\theta \;=\; \check{\mathcal{R}}^{-1}|q\rangle_\theta \otimes |p\rangle_\theta\;. \tag{4.10}$$

Here

$$\mathcal{R} \;=\; \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \;=\; \mathcal{F}^2 \tag{4.11}$$

is the universal braiding matrix of the triangular Hopf algebra structure of $\mathcal{D}_\theta$ [4, 22]. Triangularity implies that the twisted flip operator (4.5) can be written as

$$\sigma_\theta \;=\; \check{\mathcal{R}} \circ \sigma \;=\; \check{\mathcal{R}}^{-1}\,, \tag{4.12}$$

and hence braiding and transposition coincide on the twisted Fock space $\mathcal{H}^\theta$.

The commutation relations (4.10) uniquely follow from the twisting of the creation and annihilation operators $\check{a}_\theta(p), \check{a}_\theta^\dagger(p)$, which is equivalent to the morphism (2.2), and they describe multiparticle states with ordinary (untwisted) statistics. This result agrees with those of e.g. [13, 22, 44], but disagrees with results of e.g. [6]–[9] and [29] where it was claimed that the states of $\mathcal{H}^\theta$ obey twisted statistics. Our formulation is in the spirit of [21] where conventional Bose and Fermi statistics are maintained by the twisting of quantum group symmetries, and of [31] where unitary transformations of operators such as (4.5) are consistently compensated by transformations of their representations as well (as done in [46]). As we demonstrate below, it is this definition which follows from a consistent treatment of the star products of fields that deforms the tensor algebra of $\mathcal{A}_\theta$ to the braided tensor algebra corresponding to the triangular structure $\mathcal{R}$ [4, 22]. These deformations are all required for compatibility with the noncocommutative twisted coproduct Δ_θ and the action of twisted spacetime symmetries in (2.8).

4.2 Braiding of multiparticle states

We will now derive a relation between twisted quantum fields and the definition of twisted multiparticle states of $\mathcal{H}^\theta$ as *braided* tensor products of single-particle states. To understand how this definition arises, we consider the associativity law for the star product of N fields $f_a(x)$, $a = 1, \dots, N$, with respective Fourier transforms $\tilde{f}_a(p)$. By iterating the identity (2.24) we compute

$$\begin{aligned} f_1(x) \star \cdots \star f_N(x) &= \prod_{a=1}^{N} \int \frac{\mathrm{d}^d p_a}{(2\pi)^d} \, \tilde{f}_a(p_a) \ \mathrm{e}^{\,\mathrm{i}\, p_1 \cdot x} \star \cdots \star \mathrm{e}^{\,\mathrm{i}\, p_N \cdot x} \\ &= \prod_{a=1}^{N} \int \frac{\mathrm{d}^d p_a}{(2\pi)^d} \, \tilde{f}_a(p_a) \ \mathrm{e}^{\,\mathrm{i}\, p_a \cdot x} \ \exp\left(-\frac{\mathrm{i}}{2} \sum_{a<b} p_a \cdot \theta \cdot p_b \right) . \end{aligned} \tag{4.13}$$

When applied to the mode expansion (2.13) of an untwisted quantum field on $\mathcal{H}$, this relation implies that the basic multiparticle states of noncommutative quantum field theory in the basis of momentum eigenstates are given by

$$|p_1\rangle\!\rangle_\theta \otimes \cdots \otimes |p_N\rangle\!\rangle_\theta \doteq \exp\left(-\frac{\mathrm{i}}{2} \sum_{a<b} p_a \cdot \theta \cdot p_b \right) \ |p_1\rangle \otimes \cdots \otimes |p_N\rangle \ , \tag{4.14}$$

where the vectors $|p\rangle \doteq \check{a}^\dagger(p)|\Omega\rangle \in \mathcal{H}_1$ are the ordinary Fock space states. This defines a deformation of the tensor product of single-particle states to the braided (or twisted) tensor product

$$|p_1\rangle\!\rangle_\theta \otimes \cdots \otimes |p_N\rangle\!\rangle_\theta \doteq |p_1\rangle \otimes_\star \cdots \otimes_\star |p_N\rangle \ . \tag{4.15}$$

On $\mathcal{A}_\theta$, the braided tensor product is the extension of the star product (2.9) to the tensor algebra $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$, so that

$$f \otimes_\star g \ = \ \big(\mathcal{F}^{(1)} \triangleright f\big) \otimes \big(\mathcal{F}^{(2)} \triangleright g\big) \ = \ (f \otimes 1) \star (1 \otimes g) \ . \tag{4.16}$$

With this definition one has

$$(1 \otimes_\star f) \star (g \otimes_\star 1) = \big(\mathcal{R}^{(2)} \triangleright g\big) \otimes_\star \big(\mathcal{R}^{(1)} \triangleright f\big) \ . \tag{4.17}$$

In particular, one has $f\star g=\mu_0(f\otimes_\star g)$ where $\mu_0:\mathcal{A}_0\otimes\mathcal{A}_0\to\mathcal{A}_0$ is the ordinary (untwisted) pointwise multiplication on the commutative algebra $\mathcal{A}_0$. Notice that the derivation given above uses only the associativity of the star product. It therefore also applies to quantum field theories on more general noncommutative spaces, provided that one is able to define momentum eigenstates (i.e. free particles).

Consider now the indecomposable twisted state vector in $\mathcal{H}_N^\theta$ given by

$$\begin{aligned}|p_1\rangle_\theta\otimes\cdots\otimes|p_N\rangle_\theta &\doteq \check a_\theta^\dagger(p_1)\cdots\check a_\theta^\dagger(p_N)|\Omega\rangle\\ &= \check a^\dagger(p_1)\ \mathrm{e}^{\frac{\mathrm{i}}{2}\,p_1\cdot\theta\cdot\check P}\ \cdots\ \check a^\dagger(p_N)\ \mathrm{e}^{\frac{\mathrm{i}}{2}\,p_N\cdot\theta\cdot\check P}|\Omega\rangle\\ &= \check a^\dagger(p_1)\ \mathrm{e}^{\frac{\mathrm{i}}{2}\,p_1\cdot\theta\cdot(p_2+\cdots+p_N)}\\ &\quad\times\check a^\dagger(p_2)\ \mathrm{e}^{\frac{\mathrm{i}}{2}\,p_2\cdot\theta\cdot(p_3+\cdots+p_N)}\cdots\check a^\dagger(p_{N-1})\ \mathrm{e}^{\frac{\mathrm{i}}{2}\,p_{N-1}\cdot\theta\cdot p_N}\ \check a^\dagger(p_N)|\Omega\rangle\ .\end{aligned}\tag{4.18}$$

The overall exponential in (4.18) is inverse to the exponential defining the braided tensor product in (4.14). Hence the relation between twisted multiparticle states and ordinary Fock space vectors is inverse to (4.15), and one has

$$|p_1\rangle\otimes\cdots\otimes|p_N\rangle=|p_1\rangle_\theta\otimes_\star\cdots\otimes_\star|p_N\rangle_\theta\ .\tag{4.19}$$

Since the noncommutative star product may be defined in terms of the braiding of the tensor product alone, it follows that by using both braiding and twisting one cancels out the effects of noncommutativity. This is simply another version of the observations regarding the commutative dipole field operators given in section 2.2, and is the basis for the observations of [22, 44]. In particular, both formulas (4.15) and (4.19) agree with the general conclusions of [36], derived using methods of braided quantum field theory, concerning the equivalences between correlation functions of the commutative and noncommutative field theories. Notice that by employing both the braided tensor product *and* the twisted oscillators into the definition of the quantum field theory, the standard flip operators σ generate a representation of the permutation group on the resulting Fock space.

On the other hand, one could have defined the braided Fock space, i.e. the Fock space defined in terms of braided tensor products, by using twisted creation and annihilation operators with the *opposite* twist. This would be tantamount to the definitions

$$\begin{aligned}\check b_\theta(p)&\doteq\check a(p)\ \mathrm{e}^{\frac{\mathrm{i}}{2}\,p\cdot\theta\cdot\check P}\ ,\\ \check b_\theta^\dagger(p)&\doteq\check a^\dagger(p)\ \mathrm{e}^{-\frac{\mathrm{i}}{2}\,p\cdot\theta\cdot\check P}\ .\end{aligned}\tag{4.20}$$

It would manifestly reproduce all the relations for the Moyal star product of the corresponding quantum fields. We will see in the following that the Fock space formalism of this section is the only one which is compatible with duality transformations of the twisted quantum fields.

5. Twisted quantum fields in magnetic backgrounds

In this section we will describe how the previous considerations generalize to charged scalar fields in a constant background magnetic field. These models define duality-covariant noncommutative quantum field theories [32, 33]. Their infrared and ultraviolet regimes are indistinguishable due to the duality, a property which eliminates the pathologies associated to UV/IR mixing and renders the noncommutative field theory renormalizable [26]. These models naturally arise when we consider the behaviour of twisted quantum fields under duality transformations of the noncommutative spacetime, as will be further elaborated in the subsequent sections. We will find that the effective spacetime geometry underlying the twisted quantum fields in these cases is modified by the magnetic background.

5.1 Duality covariant quantum fields

We begin by deforming the algebra $\mathcal{D}_\theta$ of differential operators defined by (2.1) to an algebra $\mathcal{D}_{\theta,F}$ whose generators $\hat{x}_i, \hat{p}^j$ obey the commutation relations

$$\begin{aligned} \big[\hat{x}_i\,,\,\hat{x}_j\big] &= \mathrm{i}\,\theta_{ij}\,, \\ \big[\hat{x}_i\,,\,\hat{p}^j\big] &= \mathrm{i}\,\delta_i{}^j\,, \\ \big[\hat{p}^i\,,\,\hat{p}^j\big] &= -\,\mathrm{i}\,F^{ij}\,, \end{aligned} \tag{5.1}$$

where F^{ij} is an additional antisymmetric tensor in $GL(d,\mathbb{R})$ which plays the role of a constant background magnetic field. The operators $\hat{p}^i$ are interpreted as magnetic translation operators, which translate the coordinate operators $\hat{x}_i$ in the standard way and commute with the usual Landau hamiltonian $\frac{1}{2}\sum_i \hat{p}_i^2$ for the motion of charged particles in the magnetic background F^{ij}. The analog of the algebra morphism (2.2) is somewhat more involved in this case, and was described originally in [39]–[41] in the context of a background independent formulation of noncommutative Yang-Mills theory. Its existence is guaranteed by Darboux's theorem which implies that there is a linear transformation of operators bringing the commutation relations (5.1) into the canonical form (2.4).

Thus there exists constant antisymmetric matrices $\Lambda, \Pi \in GL(d,\mathbb{R})$ such that the differential operators

$$\begin{aligned} X_i &= \hat{x}_i + \Lambda_{ij}\;\hat{p}^j\,, \\ P^i &= \hat{p}^i + \Pi^{ij}\;\hat{x}_j \end{aligned} \tag{5.2}$$

generate the algebra $\mathcal{D}_0$. Substituting (5.2) into the canonical commutation relations (2.4) shows that Λ and Π are determined by the $d \times d$ matrix equations

$$\begin{aligned} 2\,\Lambda - \Lambda\,F\,\Lambda &= \theta\,, \\ 2\,\Pi - \Pi\,\theta\,\Pi &= F\,, \\ \Lambda\,\Pi - \Lambda\,F - \theta\,\Pi &= 0\,. \end{aligned} \tag{5.3}$$

When $F = 0$, the first and third equations give $\Lambda = \frac{1}{2}\,\theta$ and $\Pi = 0$, while the second one is then an identity. In this case (5.2) reduces to the expected mapping (2.3) in the absence of the background field. Conversely, when $\theta = 0$ one has $\Pi = \frac{1}{2}\,F$ and $\Lambda = 0$, and (5.2) is just the standard relationship between momentum and magnetic translation operators for the propagation of charged particles in a constant magnetic field. Note the perfect symmetry between position and momentum variables. This is the crux of the duality-covariant model [32]. In this case the dipole momenta P^i yield additional nonlocal field redefinitions in momentum space in terms of magnetic translations, and hence lead to a covariance under the UV/IR duality between noncommutative dipoles and elementary noncommutative fields.

It is easy to see that the equations (5.3) cannot be satisfied when $F = \theta^{-1}$. From the first two equations one shows that $F = \theta^{-1}$ if and only if $\Lambda = \theta$ and $\Pi = \theta^{-1}$, but this is inconsistent with the third equation. The case $F = \theta^{-1}$ is somewhat special and must be handled separately [39]–[41]. We shall therefore deal first with the generic case where $F \neq \theta^{-1}$.

Both the appropriate analogs of the Weyl transform and the Drinfeld twist further require commuting momentum operators which generate translations in the noncommuting coordinates $\hat{x}_i$ in the standard way. The operators

$$\tilde{P}^i = \left(\frac{1}{\mathrm{id}_d - \theta\,\Pi}\right)^i{}_j\ P^j \tag{5.4}$$

satisfy the requisite commutation relations

$$\big[\tilde{P}^i\,,\,\tilde{P}^j\big] = 0 \qquad \text{and} \qquad \big[\hat{x}_i\,,\,\tilde{P}^j\big] = \mathrm{i}\,\delta_i{}^j\ . \tag{5.5}$$

Note that $\Pi \neq \theta^{-1}$ by our assumption that $F \neq \theta^{-1}$. The twist operator defining the standard Moyal product associated to the noncommutative algebra $\mathcal{A}_\theta$ in the magnetic background is then obtained by substituting P^i with $\tilde{P}^i$ in (2.10). Thus the required abelian Drinfeld twist associated to the algebra $\mathcal{D}_{\theta,F}$ is given by

$$\tilde{\mathcal{F}} = \exp\Big(-\tfrac{\mathrm{i}}{2}\,\tilde{\theta}_{ij}\,P^i\otimes P^j\Big) \qquad \text{with} \quad \tilde{\theta} \doteq \frac{1}{\mathrm{id}_d - \Pi\,\theta}\ \theta\ \frac{1}{\mathrm{id}_d - \theta\,\Pi}\ . \tag{5.6}$$

Consider now a non-relativistic complex scalar field with the mode expansion

$$\check{\phi}(x) = \int \frac{\mathrm{d}^dk}{(2\pi)^d}\ \Big(\check{a}(k)\ \mathrm{e}^{\,-\mathrm{i}\,k\cdot x} + \check{b}^\dagger(k)\ \mathrm{e}^{\,\mathrm{i}\,k\cdot x}\Big)\ , \tag{5.7}$$

where the pair of creation and annihilation operators $(\check{a}(k), \check{a}^\dagger(k))$ and $(\check{b}(k), \check{b}^\dagger(k))$ generate two mutually commuting copies of the canonical commutation relation algebra ($2p^0 \to 1$ in (2.14)). This realizes the fields (5.7) as operator-valued distributions on the Fock space $\mathcal{H}_{\check{a}}\otimes\mathcal{H}_{\check{b}}$. The Fourier momenta k in this expansion are identified as the eigenvalues of the commuting dipole momentum operators P^i, while the coordinates x are the eigenvalues of the dipole position operators X_i. Similar expansions of commutative quantum fields in magnetic backgrounds are considered in [30].

By using the normalized translation generators (5.4) to shift the parity operator

$$\substack{\circ\\\circ}\,\delta^d(\hat{x}-\xi)\,\substack{\circ\\\circ} \doteq \mathrm{e}^{\,-\mathrm{i}\,\xi\cdot\tilde{P}}\ \substack{\circ\\\circ}\,\delta^d(\hat{x})\,\substack{\circ\\\circ}\ \mathrm{e}^{\,\mathrm{i}\,\xi\cdot\tilde{P}}\ , \tag{5.8}$$

the definition of the Weyl transform is the same as in (2.15). This is the definition appropriate to the Moyal product (2.16) defined by (5.6). To carry out the analogous calculation to that of (2.17), we eliminate the magnetic translation operators $\hat{p}^i$ in the definition of the dipole coordinates using the second equation of (5.2). Using the commutation relations (5.5) along with the fact that the matrix $(\mathrm{id}_d - \Lambda\,\Pi)\,\Lambda$ is antisymmetric, we find

$$
\begin{aligned}
\phi_p^+(X) &= \int \mathrm{d}^d\xi\ \ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot\xi}\ \check{a}(p)\ {}^{\circ}_{\circ}\,\delta^d(\xi - \hat{x} - \Lambda\cdot\hat{p})\,{}^{\circ}_{\circ} \\
&= \int \mathrm{d}^d\xi\ \int \frac{\mathrm{d}^d k}{(2\pi)^d}\ \ \mathrm{e}^{\,\mathrm{i}\,(k-p)\cdot\xi}\ \check{a}(p)\ \ \mathrm{e}^{\,-\,\mathrm{i}\,k\cdot(\hat{x}+\Lambda\cdot\hat{p})} \\
&= \check{a}(p)\ \ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot((\mathrm{id}_d-\Lambda\,\Pi)\cdot\hat{x}+\Lambda\,(\mathrm{id}_d-\theta\,\Pi)\cdot\tilde{P})} \\
&= \check{a}(p)\ \ \mathrm{e}^{\,-\,\frac{\mathrm{i}}{2}\,p\cdot(\mathrm{id}_d-\Lambda\,\Pi)\,(\Lambda\,\theta\,\Pi)\cdot p}\ \ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot\Lambda\cdot P}\ \ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot(\mathrm{id}_d-\Lambda\,\Pi)\cdot\hat{x}}\ .
\end{aligned}
\tag{5.9}
$$

This result suggests the definition of twisted oscillators

$$
\begin{aligned}
\check{a}_{\theta,F}(p) &\doteq \ \mathrm{e}^{\,-\,\frac{\mathrm{i}}{2}\,p\cdot(\mathrm{id}_d-\Lambda\,\Pi)\,(\Lambda\,\theta\,\Pi)\cdot p}\ \check{a}(p)\ \ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot\Lambda\cdot\check{P}}\ , \\
\check{b}_{\theta,F}(p) &\doteq \ \mathrm{e}^{\,-\,\frac{\mathrm{i}}{2}\,p\cdot(\mathrm{id}_d-\Lambda\,\Pi)\,(\Lambda\,\theta\,\Pi)\cdot p}\ \check{b}(p)\ \ \mathrm{e}^{\,-\,\mathrm{i}\,p\cdot\Lambda\cdot\check{P}}
\end{aligned}
\tag{5.10}
$$

and similarly for their hermitean conjugates, where

$$
\check{P}^i = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\ k^i\ \Big(\check{a}^\dagger(k)\,\check{a}(k) + \check{b}^\dagger(k)\,\check{b}(k)\Big)
\tag{5.11}
$$

is the total momentum operator on $\mathcal{H}_{\check{a}} \otimes \mathcal{H}_{\check{b}}$. These operators obey the twisted canonical commutation relations

$$
\begin{aligned}
\check{a}_{\theta,F}(p)\,\check{a}_{\theta,F}(q) &= \ \mathrm{e}^{\,2\,\mathrm{i}\,p\cdot\Lambda\cdot q}\ \check{a}_{\theta,F}(q)\,\check{a}_{\theta,F}(p)\,, \\
\check{a}_{\theta,F}(p)\,\check{a}^\dagger_{\theta,F}(q) &= \ \mathrm{e}^{\,-2\,\mathrm{i}\,p\cdot\Lambda\cdot q}\ \check{a}^\dagger_{\theta,F}(q)\,\check{a}_{\theta,F}(p) + \delta^d(p-q)\,, \\
\check{b}_{\theta,F}(p)\,\check{b}_{\theta,F}(q) &= \ \mathrm{e}^{\,2\,\mathrm{i}\,p\cdot\Lambda\cdot q}\ \check{b}_{\theta,F}(q)\,\check{b}_{\theta,F}(p)\,, \\
\check{b}_{\theta,F}(p)\,\check{b}^\dagger_{\theta,F}(q) &= \ \mathrm{e}^{\,-2\,\mathrm{i}\,p\cdot\Lambda\cdot q}\ \check{b}^\dagger_{\theta,F}(q)\,\check{b}_{\theta,F}(p) + \delta^d(p-q)\,, \\
\check{a}_{\theta,F}(p)\,\check{b}_{\theta,F}(q) &= \ \mathrm{e}^{\,2\,\mathrm{i}\,p\cdot\Lambda\cdot q}\ \check{b}_{\theta,F}(q)\,\check{a}_{\theta,F}(p)\,, \\
\check{a}_{\theta,F}(p)\,\check{b}^\dagger_{\theta,F}(q) &= \ \mathrm{e}^{\,-2\,\mathrm{i}\,p\cdot\Lambda\cdot q}\ \check{b}^\dagger_{\theta,F}(q)\,\check{a}_{\theta,F}(p)\ , \\
\check{a}^\dagger_{\theta,F}(p)\,\check{b}^\dagger_{\theta,F}(q) &= \ \mathrm{e}^{\,2\,\mathrm{i}\,p\cdot\Lambda\cdot q}\ \check{b}^\dagger_{\theta,F}(q)\,\check{a}^\dagger_{\theta,F}(p)
\end{aligned}
\tag{5.12}
$$

along with their hermitean conjugates. They generate a twisted Fock space which is no longer a free product of independent Fock spaces $\mathcal{H}^{\theta,F}_{\check{a}}$ and $\mathcal{H}^{\theta,F}_{\check{b}}$. In particular, the (untwisted) transposition of particle and antiparticle states produces nonlocal correlations.

The corresponding field operators

$$
\check{\phi}^{\theta,F}(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\ \Big(\check{a}_{\theta,F}(k)\ \ \mathrm{e}^{\,-\,\mathrm{i}\,k\cdot x} + \check{b}^\dagger_{\theta,F}(k)\ \ \mathrm{e}^{\,\mathrm{i}\,k\cdot x}\Big)
\tag{5.13}
$$

obey the dipole field relation

$$
\check{\phi}(X) = \check{\phi}^{\theta,F}\big((\mathrm{id}_d - \Lambda\,\Pi)\cdot\hat{x}\big)\ .
\tag{5.14}
$$

From (5.14) we learn that the twisted field operators together with the original star product defined by the twist element (5.6) do *not* generate a commutative algebra, due to the presence of the magnetic background $\Pi \neq 0$. Instead, the commutative dipole field algebra is obtained from the Moyal product with a background dependent redefinition of the noncommutativity parameter $\theta \mapsto \theta^F$ given by

$$\theta^F = (\mathrm{id}_d - \Pi\,\Lambda)\;\theta\;(\mathrm{id}_d - \Lambda\,\Pi)\;. \tag{5.15}$$

In particular, the twisted spacetime symmetries act on a new noncommutative algebra $\mathcal{A}_{\theta^F}$. The new twisting is a consequence of the appearence of magnetic translation operators $\hat{p}^i$ in the second line of (5.9). This is a realization of the proposal of [14] for defining twist elements associated to gauge symmetries using covariant derivatives.

Such a transformation of the moduli of the scalar field theory is anticipated by the UV/IR duality transformations [32], and it is similar to the Seiberg-Witten transformation of the magnetic field in terms of open string degrees of freedom [40], which relates commutative and noncommutative descriptions of the scalar field theory. As we will discuss further in the ensuing sections, such changes in the noncommutative geometry relate the twisted quantum field theories through duality transformations obtained by varying the two-form F. Notice as well that the braiding of multiparticle states in the twisted Fock space $\mathcal{H}^{\theta,F}$ generated by the quantum operators (5.10) is determined by yet another generically distinct noncommutativity parameter 2Λ. Thus the duality covariant field theory provides a dynamical model for the families of field theories labelled by different noncommutativity parameters in e.g. [18, 22, 25, 29], wherein the twistings of the coordinate algebra and of the quantum field operators are generically different.

Finally, we note that one can employ a more "covariant" definition of the shifted parity operator by using the noncommuting magnetic translations, obtained by replacing $\tilde{P}^i$ with $\hat{p}^i$ in (5.8). One then computes as above that the braiding of multiparticle states is determined by the noncommutativity parameter $2(\mathrm{id}_d - \Lambda\,F)\,\Lambda$, while the star product defining the commutative algebra of twisted quantum fields is determined by $(\mathrm{id}_d - F\,\Lambda)\,\theta^F\,(\mathrm{id}_d - \Lambda\,F)$. It would be interesting to investigate further the general structure of solutions to the defining matrix equations (5.3) to see if and under what conditions there exists a solution for which the braiding coincides with the twisting defined by the original Moyal geometry given by (5.6). In any case, one should realize that even in the commutative case $\theta = 0$, the quantum fields above are nonlocal in momentum space with respect to magnetic translations [30].

5.2 Dimensional reduction of self-dual quantum fields

We now consider the special choice of moduli $F = \theta^{-1}$, wherein there is no algebra morphism between $\mathcal{D}_{\theta,\theta^{-1}}$ and $\mathcal{D}_0$. This is the self-dual point wherein the noncommutative quantum field theory is *invariant* under the UV/IR duality [32]. At this point the field theory becomes an exactly solvable matrix model [33], having an infinite-dimensional $\mathrm{U}(\infty)$ symmetry acting by symplectic diffeomorphisms of the spacetime. For the real duality-invariant ϕ^4-theory without the magnetic field, in perturbation theory the beta-functions of the coupling constants vanish to all orders [17]. We will demonstrate that these special features of the self-dual point can all be understood from the fact that the effective spacetime dimension seen by twisted quantum fields is reduced to $\frac{d}{2} \doteq n$.

Generally, the differential operators

$$\hat{d}^i \doteq \hat{p}^i + \left(\theta^{-1}\right)^{ij} \hat{x}_j \tag{5.16}$$

obey the commutation relations

$$\left[\hat{d}^i\,,\,\hat{x}_j\right] \;=\; 0 \qquad \text{and} \qquad \left[\hat{d}^i\,,\,\hat{d}^j\right] \;=\; -\,\mathrm{i}\left(F-\theta^{-1}\right)^{ij}\ . \tag{5.17}$$

At the self-dual point $F=\theta^{-1}$, the operators $\hat{d}^i$ thus belong to the center of the algebra $\mathcal{A}_\theta$, which is just $\mathbb{C}$ (the constant functions). Up to an irrelevant constant shift one thus has $\hat{d}^i=0$ or

$$\hat{p}^i = -\left(\theta^{-1}\right)^{ij}\ \hat{x}_j\ . \tag{5.18}$$

This is simply the unique irreducible representation of the Heisenberg commutation relations, and there is a reduction $\mathcal{D}_{\theta,\theta^{-1}} \cong \mathcal{A}_\theta$. There are no independent momentum operators and one must instead exploit the irreducibility of the representation to build the twisted quantum states of the scalar field theory. This leads to a corresponding reduction of the Fock space at the self-dual point.

We can choose a basis of $\mathbb{R}^d$ in which the antisymmetric matrix $\theta=(\theta_{ij})$ assumes its Jordan canonical form

$$\theta = \begin{pmatrix} & -\theta_1 & & & \\ \theta_1 & & & & \\ & & \ddots & & \\ & & & & -\theta_n \\ & & & \theta_n & \end{pmatrix} \tag{5.19}$$

where $d=2n$ and $\theta_a \neq 0$ for $a=1,\dots,n$. In this basis the algebra $\mathcal{A}_\theta = \bigoplus_a \mathcal{A}_{\theta_a}(\mathbb{R}^2)$ splits into n mutually commuting blocks of noncommutative two-planes. Then the operators

$$\begin{aligned} X_a &= \hat{x}_{2a-1}\ , \\ P^a &= -\frac{1}{\theta_a}\,\hat{x}_{2a} \end{aligned} \tag{5.20}$$

for $a=1,\dots,n$ generate the canonical commutation relation algebra of $\mathcal{D}_0(\mathbb{R}^{d/2})$. Lacking a set of independent translation generators for the noncommutative space, we must use these coordinates as our set of canonically conjugate variables. Both the dipole coordinates X_a and momenta P^a are now *local* and commuting.

The corresponding twisted quantum fields are given by

$$\check{\phi}^{\theta,\theta^{-1}}(X) = \int \frac{\mathrm{d}^n k}{(2\pi)^n}\ \left(\check{a}_{\theta,\theta^{-1}}(k)\ \mathrm{e}^{\,-\,\mathrm{i}\,k\cdot X} + \check{b}^\dagger_{\theta,\theta^{-1}}(k)\ \mathrm{e}^{\,\mathrm{i}\,k\cdot X}\right) \tag{5.21}$$

where the operators $(\check a_{\theta,\theta^{-1}}(k), \check a^\dagger_{\theta,\theta^{-1}}(k))$ and $(\check b_{\theta,\theta^{-1}}(k), \check b^\dagger_{\theta,\theta^{-1}}(k))$ generate two mutually commuting copies of the undeformed canonical commutation relation algebra in $\frac{d}{2}$ dimensions. These fields behave as ordinary, commutative quantum fields. The effects of noncommutativity have been absorbed into a dimensional reduction of the effective spacetime. This is analogous to what happens in a system of charged particles constrained to lie in the lowest Landau level, where the phase space is degenerate and the wavefunctions depend on only half of the position coordinates. Untwisted noncommutative fields living in the algebra $\mathcal{A}_\theta$ can be quantized by other means [26, 32, 33]. But due to the locality of the dipole operators in this degenerate case, the twisted quantum field theory admits only a local commutative description in half the spacetime dimension. The action of twisted spacetime symmetries is truncated to a $\mathrm{Diff}(\mathbb{R}^{d/2})$ subgroup, but now there is an action of $\mathrm{U}(\infty)$ by *inner* automorphisms of the algebra $\mathcal{A}_\theta$ generating an action of the symplectomorphism group of $\mathbb{R}^d$ on untwisted noncommutative fields [34].

6. Twisted quantum fields on rational noncommutative tori

The definition (4.14) leads to the problem, in the untwisted case, of statistics of particles in the traditional approach to noncommutative quantum field theory. We noted in section 4.1 that, if any phase appearing in the braiding factor respects the condition

$$p_a \cdot \theta \cdot p_b ~=~ 2\pi\, n \qquad \text{for} \quad n \in \mathbb{Z}\,, \tag{6.1}$$

then the braiding resulting from the action of the flip operator σ on the corresponding two-particle state disappears. In this section we will extend these considerations to cases where the noncommutative field theory is defined on a torus fulfilling the more general condition

$$p_a \cdot \theta \cdot p_b ~=~ 2\pi\, h \qquad \text{for} \quad h \in \mathbb{Q}\ . \tag{6.2}$$

In this case the the braiding phase factor in $d = 2$ dimensions is of the form

$$\exp\big(2\pi\,\mathrm{i}\,\tfrac{l}{N}\,(n_1\,m_2 - m_1\,n_2)\big) \qquad \text{with} \quad l, N \in \mathbb{N}\,,\ l < N\ . \tag{6.3}$$

Quantum field theory with complex scalar fields that are adjoint sections of a gauge bundle over a noncommutative torus with rational dimensionless noncommutativity parameter is equivalent, by gauge Morita equivalence, to a quantum field theory defined on a commutative dual torus, with related moduli. In this section we will apply the standard construction of the dual field theory [2, 28, 42] to twisted quantum fields on a rational noncommutative torus and confirm that our definition of the twisted Fock space maps to the usual one of commutative quantum field theory. We consider only the two-dimensional case here in order to elucidate the braided construction of section 4 in as concise a way as possible. More general duality transformations, including the higher-dimensional cases, will be treated in depth in the next section.

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6.1 Duality transformations of quantum fields

The lattice Γ defining the original noncommutative torus is determined by the period matrix Σ and the standard square lattice of unit spacing defined by the vectors of $\tilde{\Gamma} \doteq \mathbb{Z}^2$. The period matrix may be regarded as a map

$$\big(\Sigma \,:\, \tilde{\Gamma} \,\longrightarrow\, \Gamma\big) \in GL(2,\mathbb{R}) \ . \tag{6.4}$$

The nonrelativistic quantum field on the noncommutative torus is given by the mode expansion

$$\check{\phi}(x) = \sum_{p\in\Gamma} \check{a}(p)\ \mathrm{e}^{\,-2\pi\,\mathrm{i}\,p\cdot\Sigma^{-1}\cdot x}\ , \tag{6.5}$$

where throughout we display only the positive energy particle components for brevity. This field, as a function on the covering plane, is periodic with periods defined by Σ,

$$\check{\phi}(x+\Sigma\cdot v_i) \ = \ \check{\phi}(x) \qquad \text{for} \quad i \ = \ 1,2\ , \tag{6.6}$$

where v_i are the vectors of the canonical basis of $\mathbb{Z}^2$. Hence it determines a well-defined single-valued field on the torus $\mathbb{R}^2/\Gamma$. We will regard it as a section of the trivial rank one gauge bundle over the torus.

The associated commutative dual field lives in the adjoint representation of a $\mathrm{U}(N)$ gauge group [28] and can be defined as

$$\check{\phi}^{\,\vee}(x) = \sum_{p\in\Gamma} \check{a}(p) \ \otimes\ Q^{\alpha(p)}\ P^{\beta(p)}\ \ \mathrm{e}^{\,\mathrm{i}\,\frac{\pi}{N}\,\alpha(p)\,\beta(p)}\ \ \mathrm{e}^{\,-2\pi\,\mathrm{i}\,p\cdot\Sigma^{-1}\cdot x}\ , \tag{6.7}$$

where

$$P \ \doteq \ \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \qquad \text{and} \qquad Q \ \doteq \ \begin{pmatrix} 1 & & & 0 \\ & \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{N}} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{\frac{2\pi\,\mathrm{i}\,(N-1)}{N}} \end{pmatrix} \tag{6.8}$$

are the $\mathrm{SU}(N)$ ’t Hooft shift and clock matrices which generate the $N\times N$ matrix algebra $\mathbb{M}(N,\mathbb{C})$ and obey the commutation relation

$$P\,Q = Q\,P\ \ \mathrm{e}^{\,2\pi\,\mathrm{i}/N}\ . \tag{6.9}$$

The phase factor $\exp[\,\mathrm{i}\,\frac{\pi}{N}\,\alpha(p)\,\beta(p)]$ is introduced for convenience. It entails the symmetric ordering of the matrices P and Q. We have also defined the linear functionals

$$\begin{aligned} \alpha(q) &\doteq q\cdot A\cdot v_1\ , \\ \beta(q) &\doteq q\cdot A\cdot v_2 \end{aligned} \tag{6.10}$$

with A a nondegenerate integral matrix in $\mathbb{M}(2,\mathbb{Z})$. The duality map M between the fields (6.5) and (6.7) is implemented by a correspondence between projective modules over dual tori.

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Let us consider the products of the dual field $\check{\phi}^{\vee}$. Using (6.9) they can be written as

$$
\begin{aligned}
\check{\phi}^{\vee}(x)\,\check{\phi}^{\vee}(x) &= \sum_{p,q\in\Gamma} \check{a}(p)\,\check{a}(q) \,\otimes\, Q^{\alpha(p)}\,P^{\beta(p)}\,Q^{\alpha(q)}\,P^{\beta(q)} \\
&\qquad\qquad \times\ \mathrm{e}^{\,\mathrm{i}\,\frac{\pi}{N}\,(\alpha(p)\,\beta(p)+\alpha(q)\,\beta(q))}\ \mathrm{e}^{\,-2\pi\,\mathrm{i}\,(p+q)\cdot\Sigma^{-1}\cdot x} \\
&= \sum_{p\in\Gamma} \mathrm{e}^{\,-2\pi\,\mathrm{i}\,p\cdot\Sigma^{-1}\cdot x+\frac{\mathrm{i}\,\pi}{N}\,\alpha(p)\,\beta(p)}\,Q^{\alpha(p)}\,P^{\beta(p)} \\
&\qquad\qquad \otimes\ \sum_{q\in\Gamma} \check{a}(q)\,\check{a}(p-q)\ \mathrm{e}^{\,-\,\mathrm{i}\,\frac{\pi}{N}\,\det(A)\,q\times p}\ .
\end{aligned}
\tag{6.11}
$$

The component of the product at each frequency $p\in\Gamma$ is given by the sum

$$
\sum_{q\in\Gamma} \check{a}(q)\,\check{a}(p-q)\ \mathrm{e}^{\,-\,\mathrm{i}\,\frac{\pi}{N}\,\det(A)\,q\times p}\ . \tag{6.12}
$$

If we require the duality morphism M between the fields $\check{\phi}$ and $\check{\phi}^{\vee}$ to be compatible with the products, then we have to *shrink* each polarization factor

$$
\mathrm{e}^{\,\frac{\mathrm{i}\,\pi}{N}\,\alpha(p)\,\beta(p)}\,Q^{\alpha(p)}\,P^{\beta(p)} \dashrightarrow 1\ . \tag{6.13}
$$

In this (heuristic) way we obtain the inverse image of the product

$$
M^{-1}\big[\check{\phi}^{\vee}(x)\,\check{\phi}^{\vee}(x)\big] = \sum_{p\in\Gamma} \mathrm{e}^{\,-2\pi\,\mathrm{i}\,p\cdot\Sigma^{-1}\cdot x} \sum_{q\in\Gamma} \check{a}(q)\,\check{a}(p-q)\ \mathrm{e}^{\,-\,\mathrm{i}\,\frac{\pi}{N}\,\det(A)\,q\times p}\ . \tag{6.14}
$$

This is the Moyal product $\check{\phi}(x)\star\check{\phi}(x)$ of the fields on the original torus. The noncommutativity parameter θ_{ij} can be read off from (6.14) once we restore dimensions to the wavenumbers p,q, and one has

$$
\theta_{ij} = \frac{\det(A\,\Sigma)}{2\pi\,N}\,\epsilon_{ij} = \frac{\det(A)}{N}\ \frac{\mathrm{Area}}{2\pi}\ \epsilon_{ij}\ . \tag{6.15}
$$

We thus recover the anticipated rational dimensionless noncommutativity parameter

$$
\Theta = \frac{2\pi\,\theta}{\mathrm{Area}} \tag{6.16}
$$

on the torus $\mathbb{R}^2/\Gamma$.

The dual field (6.7) is valued in the group algebra of $\mathrm{SU}(N)$, and it satisfies the periodicity conditions

$$
\begin{aligned}
P\,\check{\phi}^{\vee}(x)\,P^{-1} &= \check{\phi}^{\vee}\big(x+\tfrac{1}{N}\,(\Sigma\,A)\cdot v_1\big)\,, \\
Q^{-1}\,\check{\phi}^{\vee}(x)\,Q &= \check{\phi}^{\vee}\big(x+\tfrac{1}{N}\,(\Sigma\,A)\cdot v_2\big)\ .
\end{aligned}
\tag{6.17}
$$

By redenoting the 't Hooft matrices as

$$
\begin{aligned}
V_1 &\doteq P\,, \\
V_2 &\doteq Q^{-1}\,,
\end{aligned}
\tag{6.18}
$$

we can write the two conditions in (6.17) as the single one

$$\big(V_1^{\zeta_1}\,V_2^{\zeta_2}\big)\,\check\phi^{\,\vee}(x)\,\big(V_1^{\zeta_1}\,V_2^{\zeta_2}\big)^{-1}=\check\phi^{\,\vee}\big(x+\tfrac1N\,(\Sigma\,A)\cdot\zeta\big) \tag{6.19}$$

where $\zeta=\zeta_1\ v_1+\zeta_2\ v_2$ is a vector in $\mathbb{Z}^2$ whose components in the canonical basis are ζ_i. This shows that the operator

$$\Xi(\zeta)\doteq V_1^{\zeta_1}\,V_2^{\zeta_2} \tag{6.20}$$

implements a particular type of translation. From the relation

$$\Xi(\zeta'\,)\,\Xi(\zeta)=\Xi(\zeta+\zeta'\,)\ \ \mathrm{e}^{\,-\,\mathrm{i}\,\frac{2\pi}{N}\,\zeta_1\,\zeta'_2} \tag{6.21}$$

we deduce the commutation relations

$$\Xi(\zeta'\,)\,\Xi(\zeta)=\Xi(\zeta)\,\Xi(\zeta'\,)\ \ \mathrm{e}^{\,-\,\mathrm{i}\,\frac{2\pi}{N}\,(\zeta_1\,\zeta'_2-\zeta'_1\,\zeta_2)}\ . \tag{6.22}$$

This shows that the operators $\Xi(\zeta)$ determine a representation of the group of magnetic translations on the adjoint dual scalar field $\check\phi^{\,\vee}$ by displacement $\zeta\in\mathbb{Z}^2$. Because of the identities

$$V_i^N=\mathrm{id}_N \tag{6.23}$$

the operators $\Xi(\zeta)$ are periodic of period N in each direction. Thus they actually represent translations by vectors of $\mathbb{Z}_N^2$, which is consistent with the basic periodicity of the field $\check\phi^{\,\vee}(x)$.

We will use this periodicity constraint to regard the dual scalar field as an adjoint section of a rank N gauge bundle over a dual torus. For this, we interpret the equation (6.19) as a set of twisted boundary conditions for the dual field $\check\phi^{\,\vee}(x)$, implying that the field is periodic up to a gauge transformation on a torus defined by a possibly different lattice $\Gamma^\vee$. This would then allow us to define the dual quantum field theory on the torus $\mathbb{R}^2/\Gamma^\vee$. We thus require that there exists a period matrix $\Sigma^\vee$ and a $\mathrm{U}(N)$ gauge transformation $\Omega_\zeta(x)$ such that

$$\Omega_\zeta(x)\,\check\phi^{\,\vee}(x)\,\Omega_\zeta(x)^{-1}=\check\phi^{\,\vee}\big(x+\Sigma^\vee\cdot\zeta\big) \tag{6.24}$$

for any ζ in some set of linearly independent integral vectors $\{\zeta^{(1)},\zeta^{(2)}\}$.

Consistency of the gauge transformations in (6.24) implies that they must fulfill the cocycle conditions

$$\Omega_{\zeta'}\big(x+\Sigma^\vee\cdot\zeta\big)\circ\Omega_\zeta(x)=\Omega_\zeta\big(x+\Sigma^\vee\cdot\zeta'\,\big)\circ\Omega_{\zeta'}(x) \tag{6.25}$$

and

$$\Omega_\zeta\big(x+\Sigma^\vee\cdot\zeta\big)=\Omega_\zeta(x)\ . \tag{6.26}$$

We can write $\Omega_\zeta(x)$ in terms of magnetic translation operators, and requiring unitarity leads to the expression

$$\Omega_\zeta(x)=\ \mathrm{e}^{\,\mathrm{i}\,\zeta\cdot B\cdot x}\otimes\Xi(\zeta) \tag{6.27}$$

for some matrix $B \in \mathbb{M}(2,\mathbb{Z})$. In the commutative case at hand, we can drop the abelian phase factor in (6.27), as the cocycle conditions are automatically satisfied by a locally constant gauge transformation. This will not be the case when we will deal with a noncommutative field theory in the next section, since the star product will introduce an additional phase and one cannot disentangle $\mathrm{U}(N)$ fluxes into $\mathrm{U}(1)$ and $\mathrm{SU}(N)$ components.

Having chosen a set of linearly independent vectors $\{\zeta^{(1)},\zeta^{(2)}\}$ to represent the basis of non-contractible homology cycles of the dual torus, we obtain the fundamental domain on the covering plane $\mathbb{R}^2$ spanned by the two vectors

$$\tfrac{1}{N}\,(\Sigma\,A)\cdot\zeta^{(1)} \qquad \text{and} \qquad \tfrac{1}{N}\,(\Sigma\,A)\cdot\zeta^{(2)} \ . \tag{6.28}$$

We can restrict to the case in which the twisted periodicity (6.24) is the smallest possible, i.e. to the minimal fundamental domain. This can be translated into a condition on the allowed vectors $\zeta^{(1)}$. The fundamental magnetic translations are generated by the operators

$$T_i \doteq \Xi\big(\zeta^{(i)}\big) \tag{6.29}$$

obeying the commutation relations

$$T_i\,T_j = T_j\,T_i\ \mathrm{e}^{\,\frac{2\pi\,\mathrm{i}}{N}\,Q_{ij}}\ , \tag{6.30}$$

where the integral matrix

$$Q_{ij} \ = \ -\,\zeta^{(i)}\times\zeta^{(j)} \ = \ -\,\frac{N}{\Theta}\,\frac{\det\Sigma^\vee}{\det\Sigma}\ \epsilon_{ij} \tag{6.31}$$

represents background flux through the dual torus. The standard choice of parameters [28] is given by

$$A \ = \ \begin{pmatrix} -c & 0 \\ 0 & 1 \end{pmatrix}\ , \quad \zeta^{(1)} \ = \ \begin{pmatrix} m \\ 0 \end{pmatrix} \quad \text{and} \quad \zeta^{(2)} \ = \ \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ , \tag{6.32}$$

with the condition $\gcd(m,N)=1$ which ensures that the periodicity constraint (6.24) cannot be satisfied on a smaller torus. We then obtain for the dimensionless noncommutativity parameter

$$\Theta = -\frac{c}{N}\ \in\ \mathbb{Q} \tag{6.33}$$

and for the background magnetic flux

$$Q_{ij} = m\ \epsilon_{ij}\ . \tag{6.34}$$

6.2 Dual Fock modules

We can now compare the Fock spaces on which the quantum fields $\check\phi^{\,\vee}$ and $\check\phi$ are defined. The twisted Fock space $\mathcal{H}^\theta$ of the quantum field theory on the original noncommutative torus is defined in terms of the twisted oscillators $\check a_\theta(p), \check a_\theta^\dagger(p)$ in (2.18) (along with their negative energy antiparticle counterparts $\check b_\theta(p), \check b_\theta^\dagger(p)$ as in section 5). Using (6.7) the Fock space for the adjoint scalar field $\check{\phi}^{\vee}$ is then the module given by $\mathcal{H}^{\theta} \otimes \mathbb{C}^N$ and can be defined in terms of the oscillators

$$\begin{aligned} \check{d}(p) &\doteq \check{a}_{\theta}(p) \otimes {}^{\circ}_{\circ} Q^{\alpha(p)} \, P^{\beta(p)} {}^{\circ}_{\circ} \, , \\ \check{d}^{\,\dagger}(p) &\doteq \check{a}^{\dagger}_{\theta}(p) \otimes {}^{\circ}_{\circ} P^{-\beta(p)} \, Q^{-\alpha(p)} {}^{\circ}_{\circ} \, . \end{aligned} \tag{6.35}$$

It is easy to see that these oscillators satisfy the commutation relations

$$\check{d}(p) \, \check{d}(q) \;=\; \check{d}(q) \, \check{d}(p) \;\; \mathrm{e}^{\,-\frac{2\pi\,\mathrm{i}}{N} \, \det(A) \, p\times q} \;\; \mathrm{e}^{\,\frac{2\pi\,\mathrm{i}}{N} \, \det(A) \, p\times q} \;=\; \check{d}(q) \, \check{d}(p) \, , \tag{6.36}$$

where the first phase factor comes from the 't Hooft algebra (6.9) and the second one from the twisted oscillator algebra (2.20) (with $2p^0 \to 1$). As expected, the two factors cancel each other and one recovers the standard untwisted canonical commutation relation algebra (2.14) appropriate to the commutative dual complex scalar field theory. In other words, there is a *natural* isomorphism of Fock modules $\mathcal{H}^{\theta} \otimes \mathbb{C}^N \cong \mathcal{H}$ defined by the mapping (6.35). The phase cancellation follows from the fact that matrix multiplication with the polarization matrix ${}^{\circ}_{\circ} Q^{\alpha(p)} \, P^{\beta(p)} {}^{\circ}_{\circ}$ implements the braided tensor product in (4.14) for the present case. This matrix provides the additional twist by magnetic translation operators appropriate to the quantum field theory in the background flux Q_{ij}. The definition (6.35) agrees with the general considerations of section 5.1 and explicitly implements the proposal of [14] for gauge covariant twist elements. Completely analogous considerations also apply by fully incorporating the negative energy antiparticle components of the complex scalar fields following the treatment of section 5.

7. Twisted quantum fields on irrational noncommutative tori

When the dimensionless noncommutativity parameter of the torus is not rational-valued, the dual quantum field theory cannot be defined on a commutative torus, since via the procedure of the previous section we would obtain a parameter of the form $\Theta = \frac{\det A}{N}$. To extend the derivation of the previous section to this case, we have to resort to a dual field theory that lives on a noncommutative torus as well. Then both the original and the dual scalar field theory involve noncommutative scalar fields in the adjoint representation of respective unitary groups [2, 42]. In the dual field theory there is again a nontrivial magnetic background. Our goal is to relate the moduli of both noncommutative field theories. We will also give a more rigorous treatment of the duality transformation in the most general case, which will include the transform M of the previous section as a special case.

7.1 Duality transformations of quantum fields

We firstly need the periodic version of the Weyl transform (2.15) appropriate to fields on a d-dimensional torus with period matrix Σ. With our previous conventions one has

$$\begin{aligned} W\big[f(x)\big] \;=\; f(\hat{x}) \;&=\; \int \mathrm{d}^d\xi \; f(\xi) \; {}^{\circ}_{\circ}\, \delta(\hat{x}-\xi) \, {}^{\circ}_{\circ} \\ &=\; \int \mathrm{d}^d\xi \; f(\xi) \, \frac{1}{|\det\Sigma\,|} \sum_{p\in\Gamma} \mathrm{e}^{\,2\pi\,\mathrm{i}\,p\cdot\Sigma^{-1}\cdot(\hat{x}-\xi)} \\ &=\; \int \mathrm{d}^d\xi \; f(\xi) \, \frac{1}{|\det\Sigma\,|} \sum_{p\in\Gamma} \mathrm{e}^{\,-2\pi\,\mathrm{i}\,p\cdot\Sigma^{-1}\cdot\xi} \; {}^{\circ}_{\circ} \prod_{i=1}^{d} Y_i^{p^i} \, {}^{\circ}_{\circ} \\ &=\; \frac{1}{|\det\Sigma\,|} \sum_{p\in\Gamma} \Big(\int \mathrm{d}^d\xi f(\xi) \, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,p\cdot\Sigma^{-1}\cdot\xi} \Big) \prod_{i=1}^{d} Y_i^{p^i} \prod_{i<j} \mathrm{e}^{\,\mathrm{i}\,\pi\,\Theta_{ij}\,p^i\,p^j} \end{aligned} \tag{7.1}$$

where the plane wave coordinate operators

$$Y_i \;\doteq\; \mathrm{e}^{\,2\pi\,\mathrm{i}\,v_i\cdot\Sigma^{-1}\cdot\hat{x}} \tag{7.2}$$

obey the commutation relations

$$Y_i\,Y_j = Y_j\,Y_i \;\; \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\Theta_{ij}} \tag{7.3}$$

and Θ_{ij} is the dimensionless noncommutativity parameter

$$\Theta \doteq 2\pi \;\Sigma^{-1}\,\theta\,\big(\Sigma^{-1}\big)^{\top} \;. \tag{7.4}$$

The modular group $\mathrm{SL}(d,\mathbb{Z})$ of the noncommutative torus acts via the transformations

$$Y_i \;\longmapsto\; \prod_{k=1}^{d} Y_k^{H_{ik}} \quad \text{and} \quad \Theta \;\longmapsto\; H\,\Theta\,H^{\top} \qquad \text{for} \quad H \in \mathrm{SL}(d,\mathbb{Z}) \;. \tag{7.5}$$

This modular invariance can be used to bring the antisymmetric matrix Θ into its Jordan canonical form. The Weyl transform (7.1) takes a function $f(x)$ that actually lives on the covering space $\mathbb{R}^d$ and automatically gives an operator-valued function with periodicity given by the matrix Σ, and hence a function on the noncommutative torus deformation of $\mathbb{R}^d/\Gamma$.

Secondly, we need the generalization of the magnetic translation operators to arbitrary dimensions d. Consider the operators

$$\Xi(\zeta) \doteq \prod_{i=1}^{d} T_i^{\zeta_i} \tag{7.6}$$

where T_i are the fundamental magnetic translation operators obeying

$$T_i\,T_j = T_j\,T_i \;\; \mathrm{e}^{\,\mathrm{i}\,\frac{2\pi}{N}\,Q_{ij}} \;, \tag{7.7}$$

with Q_{ij} an antisymmetric $d \times d$ integral matrix representing the background magnetic fluxes through the two-cycles of a dual torus as before. They satisfy the commutation relations of the group of magnetic translations by $\zeta, \zeta' \in \mathbb{Z}^d$ given by

$$\Xi(\zeta)\,\Xi(\zeta') = \Xi(\zeta')\,\Xi(\zeta)\ \mathrm{e}^{\,\mathrm{i}\,\frac{2\pi}{N}\,\zeta\cdot Q\cdot\zeta'}\ . \tag{7.8}$$

Analogously to the previous section, using $\mathrm{SL}(d,\mathbb{Z})$ invariance we can fix a basis $\{\zeta^{(i)}\}_{i=1}^{d}$ of $\mathbb{Z}^d$ in which Q_{ij} assumes its normal form

$$Q = \begin{pmatrix} & -q_1 & & & \\ q_1 & & & & \\ & & \ddots & & \\ & & & & -q_n \\ & & & q_n & \end{pmatrix} \tag{7.9}$$

where $d = 2n$ and $q_1, \dots, q_n$ are integers.

At this point, we use the fact that the N-dimensional representations of the basic operators $T_i = \Xi(\zeta^{(i)})$ can be decomposed into a tensor product of several $\mathrm{SU}(N_a)$ representations each generating $\mathbb{M}(N_a, \mathbb{C})$ as an associative algebra. The ranks of the unitary groups are given by the set of integers

$$N_a \doteq \frac{N}{\gcd(N, q_a)} \tag{7.10}$$

and by the integer N_0 defined via

$$N = N_0\ \prod_{a=1}^{n} N_a\ . \tag{7.11}$$

The matrix representations are given by

$$\begin{aligned} T_{2a-1} &\doteq \mathrm{id}_{N_1} \otimes \cdots \otimes P_{N_a} \otimes \cdots \otimes \mathrm{id}_{N_n} \otimes \mathrm{id}_{N_0}\ , \\ T_{2a} &\doteq \mathrm{id}_{N_1} \otimes \cdots \otimes Q_{N_a}^{q'_a} \otimes \cdots \otimes \mathrm{id}_{N_n} \otimes \mathrm{id}_{N_0} \end{aligned} \tag{7.12}$$

with $a = 1, \dots, n$, where P_h and Q_h are the $\mathrm{SU}(h)$ 't Hooft shift and clock matrices, respectively, and we have defined the reduced fluxes

$$q'_a \doteq \frac{q_a}{\gcd(N, q_a)}\ . \tag{7.13}$$

This situation parallels the two-dimensional construction of the previous section, where we required that the fundamental domain of the covering space was the smallest one possible for the given periodicity of an adjoint scalar field. Here we see explicitly how the dimension N_a of each block in the decomposition (7.12) and the reduced fluxes q'_a are interwoven, and the requirement of minimality translates into the dimension of the representation (7.12).

We can now consider the gauge transformation operators given formally by the same formulas as in (6.27). In contrast to the commutative case, however, we cannot drop the abelian factor. We rewrite the twisted boundary conditions (6.24) using star products defined by a dual noncommutativity parameter $\theta^\vee$, which will be fixed later, to get

$$\Omega_\zeta(x) \star^\vee \check{\phi}^{\,\vee}(x) \star^\vee \Omega_\zeta(x)^{-1} = \check{\phi}^{\,\vee}\big(x + \Sigma^\vee \cdot \zeta\big) \ . \tag{7.14}$$

But now the cocycle conditions (6.26) and

$$\Omega_{\zeta'}\big(x + \Sigma^\vee \cdot \zeta\big) \star^\vee \Omega_\zeta(x) = \Omega_\zeta\big(x + \Sigma^\vee \cdot \zeta'\,\big) \star^\vee \Omega_{\zeta'}(x) \tag{7.15}$$

are no longer trivially satisfied. Instead, we obtain the condition

$$\exp\big(\,\mathrm{i}\,\zeta \cdot \big(B\,\Sigma^\vee - (B\,\Sigma^\vee\,)^\top\,\big) \cdot \zeta'\,\big)\ \exp\big(-\,\mathrm{i}\,\zeta \cdot (B\,\theta^\vee\,B^\top\,) \cdot \zeta'\,\big) = \exp\big(\,\mathrm{i}\,\tfrac{2\pi}{N}\,\zeta' \cdot Q \cdot \zeta\big) \ . \tag{7.16}$$

Since (6.26) implies $B\,\Sigma^\vee + (B\,\Sigma^\vee\,)^\top = 0$, we obtain the constraint

$$Q = -\frac{N}{2\pi}\,\big(2B\,\Sigma^\vee - B\,\theta^\vee B^\top\,\big) \tag{7.17}$$

on the matrix of magnetic fluxes through the two-cycles of the dual torus.

We have to impose the twisted boundary conditions on the magnetic background as well. Defining the background abelian gauge field

$$A_i(x) \doteq \tfrac{1}{2}\,\Phi_{ij}\,x^j \otimes \mathrm{id}_N \tag{7.18}$$

with Φ_{ij} a constant antisymmetric $d \times d$ matrix, one has the constraint

$$A_i\big(x + \Sigma^\vee \cdot \zeta\big) = \Omega_\zeta(x) \star^\vee A_i(x) \star^\vee \Omega_\zeta(x)^\dagger - \mathrm{i}\,\partial_i\Omega_\zeta(x) \star^\vee \Omega_\zeta(x)^\dagger \ . \tag{7.19}$$

This yields the consistency condition

$$\Phi \ = \ \frac{2B^\top}{\Sigma^\vee - \theta^\vee\,B^\top} \qquad \text{and} \qquad B^\top \ = \ \frac{1}{\mathrm{id}_d + \frac{1}{2}\,\Phi\,\theta^\vee}\,\frac{1}{2}\,\Phi\,\Sigma^\vee \ . \tag{7.20}$$

Substituting into (7.17) we obtain

$$\frac{2\pi}{N\ \mathrm{id}_d - Q\,\Theta^\vee}\,Q \ = \ \big(\Sigma^\vee\,\big)^\top\ \big(\Phi + \tfrac{1}{4}\,\Phi\,\theta^\vee\,\Phi\big)\ \Sigma^\vee \ = \ -\,\big(\Sigma^\vee\,\big)^\top\,F\,\Sigma^\vee \tag{7.21}$$

where

$$F_{ij} \doteq D_iA_j - \partial_jA_i \tag{7.22}$$

is the noncommutative field strength of the background gauge field A. Notice the contribution from noncommutativity coming from the gauge covariant derivative operators D_i corresponding to the background (7.18).

In order to have operator-valued functions fulfilling the constraint (7.14), we have to modify the definition of the Weyl transform (7.1) for dual fields. We can do this by introducing factors made out of the fundamental magnetic translation operators T_i, in a similar fashion to what we did for the rational case of the previous section. Thus we define the dual Weyl transform

$$W^\vee\big[f(x)\big] = \frac{1}{|\det\Sigma\,|}\,\sum_{p\in\Gamma}\,a_p(f)\ \otimes\ \prod_{i=1}^d\,T_i^{p\cdot A\cdot v_i}\ \otimes\ \prod_{i=1}^d\,Y_i^{p\cdot C\cdot v_i}\ \prod_{i<j}\ \mathrm{e}^{\,\mathrm{i}\,\pi\,\Theta_{ij}^\vee\,p^i\,p^j} \tag{7.23}$$

where A and C are nondegenerate $d\times d$ matrices with $A\in\mathbb{M}(d,\mathbb{Z})$ as before, and $a_p(f)\in\mathrm{SU}(N_0)$ replace the Fourier coefficients of the expansion of the field $f(x)$ in (7.1). We can define new plane wave coordinate operators

$$\tilde Y_i = \prod_{j=1}^d\,Y_j^{v_i\cdot C\cdot v_j}\ \otimes\ \prod_{j=1}^d\,T_j^{v_i\cdot A\cdot v_j} \tag{7.24}$$

which satisfy the commutation relations

$$\tilde Y_i\star^\vee\tilde Y_j = \tilde Y_j\star^\vee\tilde Y_i\ \ \mathrm{e}^{\,2\pi\,\mathrm{i}\,(-C\,\Theta^\vee\,C^\top+A\,Q'\,N'^{-1}\,A^\top\,)_{ij}}\ , \tag{7.25}$$

where

$$Q' = \begin{pmatrix} & -q_1' & & & \\ q_1' & & & & \\ & & \ddots & & \\ & & & & -q_n' \\ & & & q_n' & \end{pmatrix} \tag{7.26}$$

is the matrix of reduced fluxes (7.13) with respect to a fixed basis of $\mathbb{Z}^d$ and

$$N' = \begin{pmatrix} N_1 & & & & \\ & N_1 & & & \\ & & \ddots & & \\ & & & N_n & \\ & & & & N_n \end{pmatrix} \tag{7.27}$$

is the matrix of reduced ranks (7.10). One has the $d\times d$ integral matrix identity

$$L\,N'+A\,Q' = \mathrm{id}_d \tag{7.28}$$

arising from the Bézout identities for the relatively prime integers N_a and q_a'.

The commutation relations (7.25) yield an equation for the noncommutativity parameter of the original torus given by

$$\Theta = C\,\Sigma^{-1}\,\Sigma^\vee\,\Theta^\vee\,\big(\Sigma^{-1}\,\Sigma^\vee\big)^\top\,C^\top - A\,Q'\,N'^{-1}\,A^\top\ . \tag{7.29}$$

The translation generators in the magnetic background are the derivatives

$$\hat D_i\ \doteq\ \hat\partial_i-\mathrm{i}\,\big[A_i(\hat x)\,,\,-\big]\ =\ \hat\partial_i-\tfrac{\mathrm{i}}{2}\,\Phi_{ij}\,\big[\hat x^j\,,\,-\big]\ . \tag{7.30}$$

Substituting into the standard definition for translation derivations of the noncommutative torus

$$\big[\hat{D}_i\,,\,\tilde{Y}_j\big] = 2\pi\,\mathrm{i}\,\big(v_i\cdot\Sigma^{-1}\cdot v_j\big)\;\tilde{Y}_j \tag{7.31}$$

we can work out the matrix C to be

$$C = \Sigma^{-1}\,\big(\mathrm{id}_d + \tfrac12\,\theta^\vee\,\Phi\big)\;\Sigma\;. \tag{7.32}$$

Moreover, from the twisted boundary condition (7.14) and the matrix Bézout identity (7.28) we have

$$\mathrm{id}_d \;=\; -\,N'\,C\,\Sigma^{-1}\,\big(\Sigma^\vee - \theta^\vee\,B^\top\big) \;=\; -\,N'\,C\,\Sigma^{-1}\,\big(\mathrm{id}_d + \tfrac12\,\theta^\vee\,\Phi\big)\;\Sigma^\vee \tag{7.33}$$

up to an integer matrix. This leads to the relationship

$$\Sigma = \Sigma^\vee\,\big(\Theta^\vee\,Q' - N'\big) \tag{7.34}$$

between the periods of the dual noncommutative tori. Substituting into the relation (7.29) we obtain

$$\begin{aligned}\Theta &= \frac{1}{\Sigma}\,\frac{1}{\big(\mathrm{id}_d + \frac12\,\theta^\vee\,\Phi\big)^2}\,\Sigma^\vee\,\Theta^\vee\,\frac{\big(\Sigma^\vee\big)^\top}{\Sigma^\top} - A\,\frac{Q}{N}\,A^\top \\ &= -\frac{1}{\Theta^\vee\,Q' - N'}\,\frac{\Theta^\vee}{N'^\top} + \big(L - N'^{-1}\big)\,A^\top\;.\end{aligned} \tag{7.35}$$

Like the period matrix, the dimensionless noncommutativity parameter Θ is defined only modulo an integer matrix on a torus, and so we can drop the matrix $L\,A^\top$ in (7.35) to finally obtain the relationship

$$\Theta = -\frac{1}{\Theta^\vee\,Q' - N'}\,\Big(\Theta^\vee\,L^\top - A^\top\Big) \tag{7.36}$$

between the noncommutativity parameters of the dual noncommutative tori in terms of an $O(d,d;\mathbb{Z})$ transformation.

We have thereby defined an algebra $\mathcal{D}_{\Theta,F}$ generated by the operators

$$\big\{\hat{D}_i\,,\,\tilde{Y}_j\big\}\;, \tag{7.37}$$

which we have shown to fulfill the relations of the noncommutative torus with period matrix Σ given by (7.34) and with dimensionless noncommutativity parameter Θ given by (7.36). It corresponds to the original algebra of observables. This construction also shows that if we use the Weyl transform with twisted boundary conditions $W^\vee$ to define a field on a noncommutative torus with a background magnetic flux, then we can define a dual field theory on a noncommutative torus with related moduli and with no background flux, i.e. the fields on the latter torus are single-valued functions. As expected from the considerations of section 5, the momentum operators in the dual field theory are the generators $\hat{D}_i$ of the magnetic translation group corresponding to the background (7.18).

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The correspondence between the Weyl transform W given by (7.1) and the Weyl transform with magnetic background in (7.23) explains the heuristic rule (6.13) for passing to the field theory without flux. In the noncommutative field theory defined by (7.23) the components of the dual quantum field $\check{\phi}^{\vee}$ at each frequency $p \in \Gamma$ are given by the tensor product of three factors

$$\check{a}_p\big(\phi^{\vee}\big) \ \otimes\ \Xi(p\cdot A) \ \otimes\ \mathop{\circ}_{\circ} \prod_{i=1}^{d} Y_i^{p\cdot C\cdot v_i} \mathop{\circ}_{\circ} . \tag{7.38}$$

The quantum field $\check{\phi}$ in the noncommutative field theory with trivial background is reobtained from the dual field theory by identifying $\tilde{Y}_i$ as the new coordinate operators, and therefore the components of the field are given by

$$\check{a}_p(\phi) \ \otimes\ \mathop{\circ}_{\circ} \prod_{i=1}^{d} \tilde{Y}_i^{p\cdot C\cdot v_i} \mathop{\circ}_{\circ} . \tag{7.39}$$

Hence by (7.24) one has $\check{a}_p(\phi) = \check{a}_p(\phi^{\vee})$. Thus the noncommutative quantum field theory with nontrivial magnetic background has the same oscillators as the noncommutative quantum field theory without the background.

7.2 Dual Fock modules

The dual twisted quantum field theories have different oscillators, as in the presence of a background flux we must further twist by the appropriate magnetic translation operators. If the Fock module $\mathcal{H}^\theta \otimes \mathbb{C}^{N_0} \cong \mathcal{H}^\theta$ of the original twisted quantum field theory is built on twisted oscillators $\check{a}_\theta(p), \check{a}_\theta^\dagger(p)$ satisfying the algebra (2.20) (with $2p^0 \to 1$), then by (7.38) the Fock space of the dual quantum field theory is defined analogously to the rational case via the oscillators

$$\begin{aligned} \check{d}(p) &\doteq \check{a}_\theta(p) \otimes \mathop{\circ}_{\circ} \Xi(p\cdot A) \mathop{\circ}_{\circ} , \\ \check{d}^{\,\dagger}(p) &\doteq \check{a}_\theta^\dagger(p) \otimes \mathop{\circ}_{\circ} \Xi(p\cdot A)^\dagger \mathop{\circ}_{\circ} \end{aligned} \tag{7.40}$$

acting on $\mathcal{H}^\theta \otimes \mathbb{C}^N$ for $p \in \Gamma$. Using (7.29) and (2.20) the algebra of these dual creation and annihilation operators is given by

$$\begin{aligned} \check{d}(p)\,\check{d}(q) &= \check{d}(q)\,\check{d}(p)\ \ \mathrm{e}^{\,2\pi\,\mathrm{i}\,p\cdot\Theta^\vee\cdot q} , \\ \check{d}^{\,\dagger}(p)\,\check{d}^{\,\dagger}(q) &= \check{d}^{\,\dagger}(q)\,\check{d}^{\,\dagger}(p)\ \ \mathrm{e}^{\,2\pi\,\mathrm{i}\,p\cdot\Theta^\vee\cdot q} , \\ \check{d}(p)\,\check{d}^{\,\dagger}(q) &= \check{d}^{\,\dagger}(q)\,\check{d}(p)\ \ \mathrm{e}^{\,-2\pi\,\mathrm{i}\,p\cdot\Theta^\vee\cdot q} + \delta^d(p-q)\otimes \mathrm{id}_N \ . \end{aligned} \tag{7.41}$$

These commutation relations obviously reduce to the ones of section 6 in the case when $\Theta^\vee = 0$. The dual oscillators thus form a representation of the twisted canonical commutation relation algebra with noncommutativity parameter $\theta^\vee$, and hence the mapping (7.40) provides a natural isomorphism of twisted Fock modules $\mathcal{H}^\theta \otimes \mathbb{C}^N \cong \mathcal{H}^{\theta^\vee}$. Again the negative energy antiparticle components of the complex scalar fields are straightforwardly incorporated into this discussion following the treatment of section 5. We conclude that the twisted Fock module transforms covariantly under Morita duality, given our definition for the twisted oscillators as above.

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Let us cast this conclusion into a more algebraic framework which can be extended to the description of dual twisted symmetries for more general noncommutative spacetimes. Let $\mathcal{A}$ be the noncommutative torus algebra with noncommutativity parameter θ, and $\mathcal{A}^\vee$ its dual with noncommutativity parameter $\theta^\vee$. The statement that these two algebras are (strongly) Morita equivalent means that there exists an $(\mathcal{A},\mathcal{A}^\vee)$-bimodule $\mathcal{M}^\vee$ and an $(\mathcal{A}^\vee,\mathcal{A})$-bimodule $\mathcal{M}$ with the property that for any left $\mathcal{A}$-module $\mathcal{E}$, the module $\mathcal{E}^\vee = \mathcal{M}\otimes_{\mathcal{A}}\mathcal{E}$ is a left $\mathcal{A}^\vee$-module such that $\mathcal{M}^\vee\otimes_{\mathcal{A}^\vee}\mathcal{E}^\vee = \mathcal{M}^\vee\otimes_{\mathcal{A}^\vee}\mathcal{M}\otimes_{\mathcal{A}}\mathcal{E} = \mathcal{E}$. The above calculation shows that this correspondence extends to the quantum level for the field operators acting on the dual Fock modules $\mathcal{E}\otimes\mathcal{H}^\theta$ and $\mathcal{E}^\vee\otimes\mathcal{H}^{\theta^\vee}$, given our definition for braiding of multiparticle states. The bimodule property implies that the left and right actions of the algebras commute, $(f\triangleright\psi)\triangleleft f^\vee = f\triangleright(\psi\triangleleft f^\vee)$ for all $f\in\mathcal{A}$, $f^\vee\in\mathcal{A}^\vee$ and $\psi\in\mathcal{M}^\vee$. When tensored with the Fock bimodule $\mathcal{H}^\theta\otimes(\mathcal{H}^{\theta^\vee})^*$, this is the content of the correspondence between the mode expansion coefficients (7.38) and (7.39) of dual quantum fields. This equivalence also leads to a kind of duality between the Hopf algebras of twisted spacetime symmetries acting on the algebras $\mathcal{A}$ and $\mathcal{A}^\vee$. It would be interesting to understand the duality more thoroughly at this algebraic level.

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